Maximizing the Growth Rate under Risk Constraints[1]

December 3, 2008

Traian A. Pirvu
Mathematics Department
The University of British Columbia
Vancouver, BC, Canada
tpirvu@math.ubc.ca

Gordan Žitković
Department of Mathematics
University of Texas at Austin
Austin, TX, USA
gordanz@math.utexas.edu

Abstract: We investigate the ergodic problem of growth-rate maximization under a class of risk constraints in the context of incomplete, Itô-process models of financial markets with random ergodic coefficients. Including *value-at-risk* (VaR), *tail-value-at-risk* (TVaR), and *limited expected loss* (LEL), these constraints can be both wealth-dependent (relative) and wealth-independent (absolute). The optimal policy is shown to exist in an appropriate admissibility class, and can be obtained explicitly by uniform, state-dependent scaling down of the unconstrained (Merton) optimal portfolio. This implies that the risk-constrained wealth-growth optimizer locally behaves like a CRRA-investor, with the relative risk-aversion coefficient depending on the current values of the market coefficients.

Keywords: ergodic control, growth-optimal portfolio, mathematical finance, portfolio constraints, stochastic control, tail value-at-risk, value-at-risk

2002 AMS Classification: 91B30, 60H30, 60G44.

JEL classification. G10

[1] The authors would like to thank Steve Shreve for helpful ideas, engaging conversations and valuable guidance in the course of writing this paper.

This material is based upon work supported by the National Science Foundation under Grant Numbers 0103814, 0139911, and 0404682. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

1. Introduction

The problem of dynamic portfolio choice has received a great deal of attention since the seminal work of Merton [Mer69, Mer71]. It has caught the attention of both the mathematical and the financial research community because of its interesting technical aspects as well as its practical applicability. The present paper aims to contribute to both of these features by considering a constrained ergodic-control problem, where the constraint - taken directly from the everyday financial practice - exhibits an unexpected degree of structure in its interplay with the objective function. Specifically, our aim is to maximize the long-term growth rate

$$\liminf_{t\to\infty} \tfrac{1}{t}\log(X_t)$$

of the investor's wealth X_t under several *risk constraints*: regulatory agencies, as well as the internal institutional policies, often require the risk inherent in the trading strategies of investors to be carefully monitored and kept under control. Among a myriad of risk measures employed by both academics and practitioners, the most recognized one is without a doubt the Value-at-Risk (VaR) which measures the magnitude of a percentile of the loss distribution. Due to several shortcomings (e.g., lack of convexity and insensitivity to catastrophic losses), the use of VaR has recently been complemented by other measures of risk (*Tail-VaR* (TVaR), for example). This development prompted us to try to consider a large class of measures of risk (containing both Var and TVaR) and study the growth-rate maximization problem where the risk in the wealth is constrained by one of these measures.

Existing research. The study of optimal control problems where the growth rate $\frac{1}{t}\log(X_t)$ of a certain controlled quantity X_t is to be maximized goes back at least to Kelly [Kel56] and Breiman [Bre61]. While such problems have been studied in a variety of settings, we focus here on the applications in finance and economics. The earliest discrete-time results were established by Hakansson [Hak70] and Thorp [Tho71], while Karatzas [Kar89] studied the continuous-time version where the stocks follow an Itô process. Aase and Øksendal [AØ88] extend the existing results to allow stock prices to jump. Taksar, Klass, and Assaf [TKA88], Pliska and Selby [PS94] and Akian, Sulem, and Taksar [AST01] address this problem in the presence of transaction costs in a Black-Scholes model. Algoet and Cover [AC88] and Cover [Cov84, Cov91] provide algorithms for maximizing the growth rate of a portfolio in a very general discrete-time model. Jamshidian [Jam91] examines the behavior of this algorithm in a continuous-time diffusion model. In a sequence of papers, Fleming and Sheu [FS99, FS00, FS04] reformulate this problem as an infinite time horizon risk-sensitive control problem in a diffusion paradigm.

The problem of maximizing the growth rate of a portfolio is an example of an ergodic stochastic control problem. The study of ergodic control dates back to Bellman [Bel57], who considered

the discrete-time case. For the continuous-time theory, see, for example, Lasry [Las74], Tares [Tar82, Tar85], and Cox and Karatzas [CK85]. An interesting reference is a survey paper by Robin [Rob83].

From a decision-theoretic point of view, the maximization of the growth rate (which is essentially equivalent to the maximization of the expected logarithmic utility) at one point seemed as a natural choice of the objective function for money managers. However, it did not take long for the community to turn a critical eye towards the high degree of risk inherent in such strategies. To make his point perfectly clear, Samuelson [Sam79] argues in words of *literally one syllable* that maximization of non-logarithmic utilities with finite time horizons should be adopted as a more desirable goal. He says:

> *He who acts in N plays to make his mean log of wealth as big as it can be made will, with odds that go to one as N soars, beat me who acts to meet my own tastes of risk.*

but then he adds:

> *When you lose - and sure can lose - with N large, you can lose real big.*

It is, therefore, only natural that a new line of research - one attempting to subdue the excessive risk from growth-maximization - has soon emerged. Grossman and Zhou [GZ93] and Cvitanić and Karatzas [CK95] study this optimization problem under so-called "drawdown constraints", where the wealth process is never allowed to fall below a fixed fraction of its maximum-to-date, and the risky assets follow an Itô process. Closer to our setting is the work of MacLean, Sanegre, Zhao and Ziemba [MSZZ04] who consider a discrete time set-up, where the maximization of capital growth subject to Value at Risk constraint is studied by means of multistage stochastic programming. The literature on the continuous-time models with risk constraints of VaR-type is much broader. Basak and Shapiro [BS01] analyze the optimal dynamic portfolio and wealth-consumption policies of utility maximizing investors who use VaR to manage their risk exposure, in a complete-market Itô-process framework. Arguing informally, they guess the solution and discuss it without providing an existence proof. One of their (heuristic) findings is that VaR-constrained risk managers actually increase exposure to risky assets compared to the unconstrained case and, to borrow a phrase from the abstract of [BS01], *"consequently incur larger losses when losses occur"*. In order to fix this deficiency, they choose another risk measure based on the risk-neutral expectation of a loss - the *Limited Expected Loss* (LEL). A drawback of their model is that the VaR is computed in a static manner, and never reevaluated after the initial date. Emmer, Klüppelberg and Korn [ESR01] consider a dynamic model with *Capital-at-Risk* (a version of VaR) limits, in the Black-Scholes-Samuelson model. However, the assumption that portfolio proportions are held fixed during the whole investment period leads to a similar problem. Dmitrasinović-Vidović, Lari-Lavassani, Li and

Ware [DVLLLW03] extend [ESR01] to the case of time dependent, deterministic, parameters and investment strategies, where analytical formulas for the optimal strategies are obtained. Gabih, Grecksch and Wunderlich [GGW05] follow [BS01] and extend their results to cover the case of a bounded expected loss and provide detailed solutions for the class of CRRA utilities, in a constant coefficients market model. They employ the martingale method to establish the optimal portfolios under constraints and conclude with some numerical results. Gundel and Weber [GW06] analyze optimal portfolio choice of utility maximizing agents in a general continuous-time financial market model under a joint budget and the downside risk constraint measured by an abstract convex risk measure (the VaR constraint used in [BS01] is a particular type of a downside risk constraint and it can be reformulated in the language of translation invariant risk measures). The utility maximization problem under these constraints is solved in closed form, and the conditions under which the the constraints are binding are determined.

An axiomatic approach to risk-measurement has started with the seminal paper of Artzner, Delbaen, Eber and Heath [ADEH], where four simple postulates (to be satisfied by any risk measures) are proposed. The resulting functionals are termed *coherent risk measures*. Föllmer and Schied [FS04a] relax one of the axioms of [ADEH] and obtain a more general notion of a *convex risk measures*. Jaschke and Küchler investigate the properties of convex risk measures in [JK01]. All of the work mentioned above assumed static modeling framework. The literature on dynamic risk measures (where a temporal component is added) is relatively new and we only mention a small portion of the existing research: in [ADEHK02] and [ADEHK04], Artzner, Delbaen, Eber, Heath and Ku construct coherent risk measures on stochastic processes rather than on random variables. Wang [W03] considers a set of axioms for dynamic risk measures and analyzes the class of measures satisfying his axioms. Delbaen, Cheridito and Kupper [DCK04] investigate the properties of risk measures defined over stochastic processes. Another approach to modeling of risk constraints - also developed with the intention of going beyond the static formulation and building on the work of [BS01] - was introduced by Cuoco, He and Issaenko [CHI07]. A more realistic, dynamically-consistent model of optimal behavior of a trader subject to risk constraints is presented here: the authors assume that the risk in the trading portfolio is reevaluated dynamically, using the current information. Hence, the trader must continuously monitor his/her trading strategy in order to honor the risk limits at every instant. Another assumption made in [CHI07] is that when assessing the risk of a portfolio, the distribution of the portfolio composition is kept unchanged over a given horizon τ (more precisely, the relative exposures to different assets are kept unchanged). In other words, the model outlaws those trading strategies which at any point t in time, if kept constant over the time interval $[t, t+\tau]$, would result in a loss whose VaR is below a given threshold. The authors perform an analogous analysis with VaR replaced by TVaR, and establish that it is possible to identify a dynamic VaR risk limit equivalent to a given TVaR risk limit. Finally, they conclude

that that the risk exposure of a trader subject to VaR or TVaR risk limits is always lower than that of an unconstrained trader. We should note that, while this paper makes a very significant contribution to modeling of risk constraints, it limits its scope to a multivariate Black-Scholes markets. Relaxing this condition is one of the main motivations for the research that lead to the present paper.

Cuoco and Liu [CL03] study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates. For a market with constant price coefficients, they show that optimal portfolios display a local three-fund property. Leippold, Trojani and Vanini [LVT02] analyze VaR-based regulation rules and their possible distortion effects on financial markets in the setting of diffusion processes. They show that in partial equilibrium the effectiveness of VaR regulation is closely linked to the *leverage effect* - the tendency of volatility to increase when the prices decline. Berkelaar, Cumperayot and Kouwenberg [BCK05] study the effect of VaR-based risk management on asset prices, (modelled as Itô processes) and the volatility smile. They look at an equilibrium model where a portion of the agents are constrained with VaR. It turns out that in equilibrium VaR reduces market volatility, but in some cases raises the probability of extreme losses.

In [Yiu04], the author considers an optimal investment problem, where an agent maximizes utility of his/her intertemporal consumption over a period of time under a dynamic VaR constraint. A numerical method is proposed to solve the corresponding HJB-equation. He finds that, under the optimal strategy, the investment in risky assets is reduced by the VaR constraint. Atkinson and Papakokinou [AP05] derive the solution to the optimal portfolio and consumption problem subject to CaR (*Capital-at-Risk*) and VaR constraints by using stochastic dynamic programming. In both [Yiu04] and [AP05] the strong assumption of constant market coefficients is imposed.

Our contributions. In this work we follow the approach of [CHI07] and impose dynamic risk constraints of the VaR-type. Unlike [CHI07], we maximize the long-term (ergodic) growth rate of the accumulated wealth. Moreover, our model allows market coefficients (the stock price return and volatility) to be random processes, assumed to satisfy a mild ergodicity condition, but without any restriction on the completeness of the resulting market. In addition to the constant-coefficient models, our set-up allows for a wide range of stochastic-volatility and seasonally-varying models.

Consequently, the risk measurement on the time interval $[t, t+\tau]$ is performed under the assumption that the market coefficients, as well as portfolio proportions, are held constant at their value at time t. While, for the sake of simplicity, our risk constraint is taken to be either VaR, TVaR or LEL, all our results hold under a more general class of risk measures, expressible as deterministic functions of two "sufficient statistics": *portfolio return* and *portfolio volatility.* Furthermore, we differentiate between two different risk-limit implementations - relative and absolute (depending

on whether the risk is measured as a percentage of current wealth, or in dollar terms). In the latter case, the constraints become wealth- (state-) dependent and the agent finds him-/herself in an interesting predicament - should he/she maximize the current growth rate of wealth, or act more conservatively and thus face more favorable constraints in the future. This raises the complexity level of the problem considerably and requires a delicate mathematical analysis, the final conclusion of which is that nothing can be gained by waiting. More precisely, the structure of the aforementioned wealth-dependent case is such that the constraints are not binding, as long as the wealth is below a certain level. Once the wealth gets above that level, the constraints become binding and the set of admissible portfolios reduces as wealth accumulates. In the limit as wealth approaches infinity, the constraints shrink and approach the *limiting constraint set* (which still depends on the market coefficients, time, and the current state of the world). One of optimal strategies we identify can be described as follows: pretend that the limiting constraint is imposed from the start and simply project (under a specific metric) the unconstrained optimal portfolio (the *Merton proportion process*) onto it. Alternatively, projecting the Merton proportion process onto the current constraint set leads to the same ergodic behavior.

In the relative case, we show that the projection of the Merton proportion onto the (current) constraint set describes the optimal behavior in both ergodic, and the finite-horizon cases.

Thanks to the special structure of the constraints, the projection of the Merton proportion onto the constraint is collinear with the origin and Merton proportion itself. This fact is the key to the success of our analysis and sheds new light on the reasons why VaR constraints, coupled with the growth-rate maximization, leads to such agreeable results. Moreover, the ratio between the norm of the projection and the norm of the Merton proportion can be interpreted as the reduction in risk-exposure of the constrained agent (compared to the unconstrained one). This number will follow a random process β_t, thus making our agent act locally as if he/she is a CRRA-utility maximizer with the coefficient of relative risk aversion depending on the current market conditions. Interestingly, we show that β_t is a nonlinear deterministic function δ of the norm of the Merton proportion process only. Furthermore, the value of the optimal growth-rate of wealth can be obtained by integrating the real function $x^2\delta(x)$ against the invariant measure of the norm of the Merton proportion process.

Organization of the paper and some remarks on notation and terminology. The reminder of this paper is organized as follow. In section 2 we describe the financial market model, the risk measures and the constraint sets. Section 3 contains the main results, and Section 4 develops the proof of the main theorem through a number of auxiliary results. The paper ends with an Appendix containing some technical results.

All random processes and random fields in the paper possess the degree of measurability sufficient for all the operations preformed on them. We do not mention, or check, this fact in the main body,

leaving the standard proofs to the interested reader. Occasionally, a phrase like "pick a typical $\omega \in \Omega$" will be used. It will mean that all the previous statements, proven to hold a.s., are assumed to hold for this particular realization $\omega \in \Omega$.

A stochastic processes $\{X_t\}_{t\in[0,\infty)}$ will usually be denoted simply by X_t (or even X), and the elements of $\mathbb{R}^n$ or $\mathbb{R}^m$ will be interpreted as column vectors in the relevant contexts.

2. Model Description and Problem Formulation

2.1. **The Financial Market.** Our model of a financial market, based on a a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ satisfying the usual conditions, consists of $n+1$ assets. The first one, $\{S_0(t)\}_{t\in[0,\infty)}$, is a *riskless bond* with a strictly positive constant interest rate $r > 0$. The remaining n are referred to as *stocks*, and are modelled by an n-dimensional Itô-process $\{\boldsymbol{S}(t)\}_{t\in[0,\infty)} = \{(S_i(t))_{i=1,\dots,n}\}_{t\in[0,\infty)}$. The dynamics of their evolution is determined by the following stochastic differential equations in which $\{\boldsymbol{W}(t)\}_{t\in[0,\infty)} = \{(W_i(t))_{i=1,\dots,m}\}_{t\in[0,\infty)}$ is an m-dimensional standard Brownian motion:

$$\left.\begin{aligned} &dS_0(t) = S_0(t) r\, dt \\ &dS_i(t) = S_i(t)\Big(\alpha_i(t)\, dt + \sum_{j=1}^{m} \sigma_{ij}(t)\, dW_j(t)\Big),\ i = 1,\dots,n, \end{aligned}\right\}, t \in [0,\infty), \tag{2.1}$$

where $\{\boldsymbol{\alpha}(t)\}_{t\in[0,\infty)} = \{(\alpha_i(t))_{i=1,\dots,n}\}_{t\in[0,\infty)}$ is an $\mathbb{R}^n$-valued *mean rate of return* processes, and $\{\boldsymbol{\sigma}(t)\}_{t\in[0,\infty)} = \{(\sigma_{ij}(t))_{i=1,\dots,n}^{j=1,\dots,m}\}_{t\in[0,\infty)}$ is an $n \times m$-matrix-valued *variance-covariance* process. In order for the equations in (2.1) to be well-defined, we impose the following regularity conditions on the coefficient processes $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\sigma}(t)$:

Assumption 2.1. All the components of the processes $\{\boldsymbol{\alpha}(t)\}_{t\in[0,\infty)}$ and $\{\boldsymbol{\sigma}(t)\}_{t\in[0,\infty)}$ are cáglád (left-continuous with right limits).

Remark 2.2.

(1) The cáglád requirement from Assumption 2.1 is used in several different ways in this paper. First, it ensures local boundedness, a property needed in several parts of the proof of the main result. Second, it is necessary for the standard SDE theory (see Lemma 4.8 below) to be applicable. Finally, it directly implies the following integrability condition

$$\sum_{i=1}^{n} \int_0^t |\alpha_i(u)|\, du + \sum_{i=1}^{n}\sum_{j=1}^{m} \int_0^t \sigma_{ij}(u)^2\, du < \infty, \text{ for all } t \in [0,\infty), \text{ a.s.}$$

(2) Further distributional restrictions will be imposed on $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\alpha}(t)$ in the sequel (the impatient reader is invited to peek ahead to Assumption 2.5).

(3) Working with multidimensional stock-prices processes is of fundamental importance for the understanding of the full scope of our results. In order to simplify the presentation, we introduce several notational shortcuts for ordinary and stochastic integrals of vector- or matrix-valued processes; for an integrable $\mathbb{R}^m$-valued process $\boldsymbol{\rho}(t) = (\rho_i(t))_{i=1,\dots,n}$, and a sufficiently regular $\mathbb{R}^m$-valued process $\boldsymbol{\pi}(t) = (\pi_j(t))_{j=1,\dots,m}$ we write

$$\int_0^t \boldsymbol{\rho}(u)\,du \triangleq \sum_{i=1}^n \int_0^t \rho_i(u)\,dt, \quad \int_0^t \boldsymbol{\pi}(t)\,d\boldsymbol{W}(t) \triangleq \sum_{j=1}^m \int_0^t \pi_j(t)\,dW_j(t).$$

2.2. **Trading strategies and wealth.** Actions of an investor in the market are modelled by the proportions of current wealth her/she invests in various assets. Specifically, we have the following formal definition.

Definition 2.3. An $\mathbb{R}^n$-valued stochastic process $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)} = \{(\zeta_i(t))_{i=1,\dots,n}\}_{t\in[0,\infty)}$ is called an *admissible portfolio-proportion process* if it is progressively measurable and satisfies

$$\int_0^t \left|\boldsymbol{\zeta}^T(u)(\boldsymbol{\alpha}(u) - r\mathbf{1})\right|\,du + \int_0^t ||\boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(u)||^2\,du < \infty, \text{ a.s., for all } t \in [0,\infty), \tag{2.2}$$

where, as usual, $\mathbf{1} = (1,\dots,1)^T$ is an n-dimensional column vector all of whose coordinates are equal to 1, and $||\boldsymbol{x}|| = \sqrt{\sum_{j=1}^m x_j^2}$ is the standard Euclidean norm of a vector $\boldsymbol{x} = (x_j)_{j=1,\dots,m} \in \mathbb{R}^m$.

Given a portfolio-proportion process $\boldsymbol{\zeta}(t)$, we interpret its n coordinates as the proportions of the current wealth $X^{\boldsymbol{\zeta}}(t)$ invested in each of n stocks. In order to remain self-financing, the left-over wealth $X^{\boldsymbol{\zeta}}(t)(1-\sum_{i=1}^n \zeta_i(t))$ is assumed to be invested in the riskless bond $S_0(t)$. Of course, if this quantity is negative, we are effectively borrowing at the rate $r > 0$. We stress that no short-selling restrictions are imposed, meaning that the proportions $\zeta_i(t)$ are allowed to be negative. Therefore, the equation governing the evolution of the total wealth $\{X^{\boldsymbol{\zeta}}(t)\}_{t\in[0,\infty)}$ of the investor using the portfolio-proportion process $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)}$ is given by

$$\begin{aligned} dX^{\boldsymbol{\zeta}}(t) &= X^{\boldsymbol{\zeta}}(t)\Big(\boldsymbol{\zeta}^T(t)\boldsymbol{\alpha}(t)\,dt + \boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\Big) + \Big(1-\boldsymbol{\zeta}^T(t)\mathbf{1}\Big)X^{\boldsymbol{\zeta}}(t)r\,dt \\ &= X^{\boldsymbol{\zeta}}(t)\Big((r+\boldsymbol{\zeta}^T(t)\boldsymbol{\mu}(t))\,dt + \boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\Big), \end{aligned} \tag{2.3}$$

where $\{\boldsymbol{\mu}(t)\}_{t\in[0,\infty)} = \{(\mu_i(t))_{i=1,\dots,n}\}_{t\in[0,\infty)}$, with $\mu_i(t) = \alpha_i(t) - r$ for $i = 1,\dots,n$, is the vector of *excess rates of return*. Under regularity conditions (2.2) imposed on $\boldsymbol{\zeta}(t)$ above, (2.3) admits a unique strong solution given by the explicit expression

$$X^{\boldsymbol{\zeta}}(t) = X(0)\exp\left(\int_0^t \Big(r + \boldsymbol{\zeta}^T(u)\boldsymbol{\mu}(u) - \tfrac{1}{2}||\boldsymbol{\zeta}^T(u)\boldsymbol{\sigma}(u)||^2\Big)\,du + \int_0^t \boldsymbol{\zeta}^T(u)\boldsymbol{\sigma}(u)\,d\boldsymbol{W}(u)\right), \tag{2.4}$$

The initial wealth $X^{\boldsymbol{\zeta}}(0) = X(0) \in (0,\infty)$, is considered a primitive of the model, and will thus be considered arbitrary but fixed throughout the paper. In particular, it will not vary with the choice of the investment strategy $\boldsymbol{\zeta}$.

2.3. **Some useful notation.**

2.3.1. *Functions $\tilde{Q}$ and Q.* The expression appearing inside the first integral in (2.4) above will be important enough in the sequel to warrant its own notation; the affine-quadratic function $\tilde{Q}:\mathbb{R}^2\to\mathbb{R}$ is defined as

$$\tilde{Q}(\zeta_{\boldsymbol{\mu}},\zeta_{\boldsymbol{\sigma}}) = r + \zeta_{\boldsymbol{\mu}} - \tfrac{1}{2}\zeta_{\boldsymbol{\sigma}}^2, \tag{2.5}$$

so that the aforementioned expression becomes $\tilde{Q}(\boldsymbol{\zeta}^T(t)\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)||)$. Oftentimes, the dependence of the drift of the process $\log(X^{\boldsymbol{\zeta}}(t))$ on the choice of the instantaneous portfolio-proportion $\boldsymbol{\zeta}$ will be important. It proves useful to define the random field $Q:\Omega\times[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ by

$$Q(t,\boldsymbol{\zeta}) = \tilde{Q}(\boldsymbol{\zeta}^T\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)||). \tag{2.6}$$

In the new notation, the process $Y^{\boldsymbol{\zeta}}(t) = \log(X^{\boldsymbol{\zeta}}(t))$ evolves according to the following simple dynamics

$$dY^{\boldsymbol{\zeta}}(t) = Q(t,\boldsymbol{\zeta}(t))\,dt + dM^{\boldsymbol{\zeta}}(t),\ t\in[0,\infty), \tag{2.7}$$

where the local martingale $\{M^{\boldsymbol{\zeta}}(t)\}_{t\in[0,\infty)}$ is defined by

$$M^{\boldsymbol{\zeta}}(t) = \int_0^t \boldsymbol{\zeta}^T(u)\boldsymbol{\sigma}(u)\,d\boldsymbol{W}(u),\ t\in[0,\infty). \tag{2.8}$$

It is clear from the expression (2.4) above that the drift of $X^{\boldsymbol{\zeta}}(t)$ depends on the $\mathbb{R}^n$-dimensional process $\boldsymbol{\zeta}(t)$ only through two "sufficient statistics"

$$\zeta_{\boldsymbol{\mu}}(t) \triangleq \boldsymbol{\zeta}^T(t)\boldsymbol{\mu}(t),\ \text{and}\ \zeta_{\boldsymbol{\sigma}}(t) \triangleq ||\boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)||. \tag{2.9}$$

They will be referred to in the sequel as **portfolio rate of return** and **portfolio volatility**, respectively.

2.3.2. *The Merton-proportion process.* In order for the definition of the Merton-proportion process to make sense, we impose the following mild condition on the variance-covariance process $\boldsymbol{\sigma}(t)$.

Assumption 2.4. The matrix $\boldsymbol{\sigma}(t)$ has independent rows for all $t\in[0,\infty)$, a.s.

The financial meaning of Assumption 2.4 is quite simple - it precludes different stocks from having the same diffusion structure. Otherwise, the market would either allow for arbitrage opportunities or redundant assets would exist. The first consequence of this assumption is that $n\leq m$ - the number of risky assets does not exceed the number of "sources of uncertainty". Also, the inverse $(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t))^{-1}$ is easily seen to exist and so the equation

$$\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t)\boldsymbol{\zeta}_M(t) = \boldsymbol{\mu}(t), \tag{2.10}$$

uniquely defines a cáglád stochastic process $\{\boldsymbol{\zeta}_M(t)\}_{t\in[0,\infty)}$, termed the *Merton-proportion process*. It has the pleasant property that (in the absence of portfolio constraints), the growth-rate- or log-optimizing investor would invest in the market exactly using the components of $\boldsymbol{\zeta}_M(t)$ as portfolio proportions (see [KS98]).

2.3.3. *A metric-valued process.* Finally, the fact that the rows of $\boldsymbol{\sigma}(t)$ are independent easily leads to the fact that the random field $\{d_{\boldsymbol{\sigma}(t)}(\cdot,\cdot)\}_{t\in[0,\infty)}$ given by

$$d_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = ||\boldsymbol{\sigma}(t)^T(\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2)||, \text{ for } \boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2 \in \mathbb{R}^n. \tag{2.11}$$

is a metric on $\mathbb{R}^n$ for all $t\in[0,\infty)$, a.s.

2.4. **The Ergodic Assumption.** Since we are dealing with a stochastic control problem of an ergodic type, we impose an ergodicity requirement on the coefficients $\boldsymbol{\mu}(t)$ and $\boldsymbol{\sigma}(t)$ driving the financial market. It turns out, somewhat surprisingly, that we only need to deal with a combination of two - a real valued process related to the Merton-proportion process defined above in (2.10). Specifically, we impose the following assumption.

Assumption 2.5 (Ergodicity of the Merton-proportion process)**.** The process $\{||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||\}_{t\in[0,\infty)}$ is ergodic in the sense that for each non-negative continuous function $\varphi:[0,\infty)\to\mathbb{R}$ satisfying $\sup_{x\in\mathbb{R}}\frac{\varphi(x)}{1+x^2}<\infty$ there exists an $\mathcal{F}_\infty$-measurable, finite random variable $Z(\varphi)$ such that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi(||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||)\,du = Z(\varphi), \text{ a.s.} \tag{2.12}$$

Remark 2.6. Multiplying both sides of (2.10) by $\boldsymbol{\zeta}_M^T(t)$ from the left shows that $||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||^2 = \boldsymbol{\zeta}_M^T(t)\boldsymbol{\mu}(t)$, so the Assumption 2.5 can be formulated equivalently in terms of the process $\boldsymbol{\zeta}_M^T(t)\boldsymbol{\mu}(t)$. This fact will be useful in some proofs in the sequel.

Example 2.7. Assumption 2.5 is the only non-trivial condition imposed on the market coefficients $\boldsymbol{\mu}(t)$ and $\boldsymbol{\sigma}(t)$, and, therefore, examples to illustrate its restrictiveness are needed. Two important classes of financial models satisfying it are presented below:

(1) When $\boldsymbol{\mu}(t)$ and $\boldsymbol{\sigma}(t)$ are deterministic constants one can easily see that Assumption 2.5 is trivially satisfied. More generally, our framework incorporated deterministic processes $\boldsymbol{\mu}(t)$ and $\boldsymbol{\sigma}(t)$ which exhibit enough periodic behavior in order for the averages introduced in (2.12) to be convergent. Deterministic coefficients of this type are used in models of seasonally-sensitive assets.
(2) A class of stochastic-volatility models also complies with Assumption 2.5. Indeed, following [FHH03] and [FT02], let us consider the special case of our model in which $n=1$, $m=2$

and

$$\mu_1(t) = \mu \in \mathbb{R},\ \sigma_{11}(t) = \rho\,\Sigma(V(t)),\ \text{and}\ \sigma_{12}(t) = \sqrt{1-\rho^2}\,\Sigma(V(t)),$$

where, the state process $\{V(t)\}_{t\in[0,\infty)}$ is given by

$$dV(t) = \nu(\overline{V} - V(t))\,dt + dW_2(t). \tag{2.13}$$

Here $\rho \in [-1,1]$ is the correlation coefficient, $\nu > 0$ is the rate of mean reversion, $\overline{V} > 0$ is the mean-reversion level, $\mu \in \mathbb{R}$ is the mean rate of return of the risky asset, and the function $\Sigma : [0,\infty) \to [0,\infty)$ transforms the state process $v(t)$ into asset volatility $\Sigma(V_t)$. We assume that Σ is a continuous function bounded both from above and away from zero. Under these conditions, the volatility $\Sigma(V(t))$ inherits the mean-reversion property of $V(t)$, reverting to level $\Sigma(\overline{V})$. In this model $||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||^2 = \mu^2\Sigma(V(t))^{-2} = g(V_t)$ for some bounded continuous function $g : [0,\infty) \to \mathbb{R}$. The processes V_t, being a one-dimensional Ornstein-Uhlenbeck process, has a finite invariant measure $\gamma(dx)$ which is Gaussian. In fact, $\gamma(dx) = \sqrt{\frac{\nu}{\pi}}e^{-\nu(x-\overline{V})^2}dx$. By $Theorem$ 3.1 in [Kha60], for any measurable function φ integrable with respect to $\gamma(dx)$, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi(V(u))\,du = \int_{-\infty}^{\infty}\varphi(x)\gamma(dx) < \infty.$$

Therefore, Assumption 2.5 holds, with $Z(\varphi) = \int_{-\infty}^{\infty}\varphi(x)\gamma(x)\,dx$.

Remark 2.8.

(1) The ergodicity assumption stated above is standard in control problems with ergodic objective criterion. A typical ergodic process used in practice is very much like the one appearing in the stochastic-volatility model in the Example 2.7, (2) above — a deterministic function of a diffusion process with a stationary distribution. While the financial models of the asset prices will not have the ergodic property, it is hard to immagine realistic models of appreciation rates or the volatility matrix which are *not* ergodic. Indeed, from the economic point of view, the lack of the ergodic structure in those processes would imply strong confidence of the modeller in the lack of any kind of equilibrium in the very-long term behaviour of the financial system under consideration.

(2) The random variable $Z(\varphi)$ is, in fact, a deterministic constant throughout Example 2.7 above. There is, however, a class of realistic situations in which it will be a true random variable. Imagine a situation in which, from the start, the market volatility is known to be as in part (2) of Example 2.7, but the mean-reversion level $\bar{V}$ is unknown, with the a-priory distribution $\nu(dx)$, which is independent of all the other sources of uncertainty in

the model. In that case, the random variable $Z(\varphi)$ will be truly random (except for special choice of the function f) and given by

$$Z(\varphi)(\omega) = \int_{-\infty}^{\infty} \varphi(x) \sqrt{\frac{\delta}{\pi}} e^{-\delta(x-\overline{V}(\omega))^2} dx.$$

More complicated cases where $\bar{V}$ is not independent of the other sources of uncertainty can be envisioned. Those situations will lead to random limits $Z(\varphi)$, but will not correspond to the simple mixtures of different stochastic volatility models any more.

2.5. **Portfolio constraints.** Having introduced the financial market, we turn to the specification of the portfolio constraints which limit the investor's behavior in each instant. We start with an abstract description of the form of these constraints and continue to present several special cases dealing with realistic risk-limits. One of the features in which our framework differs from the majority of existing work is that the set of allowable portfolio proportions depends not only on the current market conditions ($\boldsymbol{\mu}(t)$ and $\boldsymbol{\sigma}(t)$) but also on the current level of the investor's wealth $X^{\boldsymbol{\zeta}}(t)$.

Definition 2.9. A **portfolio-constraint correspondence** is a family of $(x, \boldsymbol{\mu}, \boldsymbol{\sigma}) \mapsto F(x, \boldsymbol{\mu}, \boldsymbol{\sigma}) \subseteq \mathbb{R}^m$ of subsets of $\mathbb{R}^n$ with the property that there exist two functions $f : \mathbb{R} \times [0, \infty) \to \mathbb{R} \cup \{\infty\}$ and $h : (0, \infty) \to \mathbb{R}$ such that

$$F(x, \boldsymbol{\mu}, \boldsymbol{\sigma}) = F_{(f,h)}(x, \boldsymbol{\mu}, \boldsymbol{\sigma}) = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^m \ : \ f(\boldsymbol{\zeta}^T \boldsymbol{\mu}, ||\boldsymbol{\zeta}^T \boldsymbol{\sigma}||) \leq h(x) \right\}.$$

The function f is assumed to satisfy the following conditions:

(1) $f \in C^1(\mathbb{R} \times [0, \infty))$ is jointly convex.

(2) For each $(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \in \mathbb{R} \times [0, \infty)$, the sections $f(\zeta_{\boldsymbol{\mu}}, \cdot)$ and $f(\cdot, \zeta_{\boldsymbol{\sigma}})$ are (respectively) strictly increasing and decreasing.

(3) $f(0, 0) < 0$ and there exist constants $\kappa_i > 0$, $i \in \{1, 2, 3\}$ such that for all $(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \in \mathbb{R} \times [0, \infty)$

$$f(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \geq \kappa_1 \zeta_{\boldsymbol{\sigma}}^2 - \kappa_2 \zeta_{\boldsymbol{\mu}} - \kappa_3 \tag{2.14}$$

For the function h, we require either of the following two sets of assumptions:

(A) $h(x) = c$ for some $c \in (0, \infty)$ and all $x > 0$, or

(B)

(B.1) $h(\exp(\cdot))$ is convex,

(B.2) there exists $x_0 > 0$ such that $h(x) = +\infty$ for $x \leq x_0$,

(B.3) $h(\cdot)$ is finite, strictly decreasing and continuously differentiable on (x_0, ∞), and

(B.4) $\lim_{x \searrow x_0} h(x) = +\infty$, $\lim_{x \to \infty} h(x) = 0$.

The constraints are said to be **relative** in the case (A), and **absolute** if (B) holds.

In notational analogy with the function $\tilde{Q}$ and the random field Q (defined in (2.5) and (2.6)), for each portfolio-constraint correspondence $F = F_{(f,h)}$ as in Definition 2.9, we define a random set-valued field (random correspondence field) $\tilde{F} : \Omega \times [0,\infty) \times \mathbb{R} \to 2^{\mathbb{R}^n}$ by

$$\tilde{F}(t,x) = \tilde{F}_{(f,h)}(t,x) = F(x, \boldsymbol{\mu}(t), \boldsymbol{\sigma}(t)). \tag{2.15}$$

This parallel notation will be very useful in the later sections of the manuscript.

Imposing a portfolio constraint dynamically leads to the following definition of the set of admissible portfolio processes.

Definition 2.10. An $\mathbb{R}^n$-valued process $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)}$ is said to be (f,h)**-admissible** if it is an admissible portfolio-proportion process (in the sense of Definition 2.3) and

$$\boldsymbol{\zeta}(t) \in F_{(f,h)}(t, X^{\boldsymbol{\zeta}}(t)) = \tilde{F}_{(f,h)}(X^{\boldsymbol{\zeta}}(t), \boldsymbol{\mu}(t), \boldsymbol{\sigma}(t)),\ \text{for all } t \in [0,\infty), \text{ a.s.},$$

where the dynamics of the process $\{X^{\boldsymbol{\zeta}}(t)\}_{t\in[0,\infty)}$ is given in (2.3) and (2.4). The set of all admissible portfolio-proportion processes $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)}$ will be denoted by $\mathcal{A}_{(f,h)}$, or simply $\mathcal{A}$, when no confusion can arise.

2.6. **Examples of portfolio constraints.** The discussion that follows aims to show that a number of risk-based portfolio constraints used in the literature allows a formulation from Definition 2.9.

2.6.1. *Projected distribution of wealth.* For the purposes of risk measurement, it is a common practice to use an approximation of the distribution of the investor's wealth at a future date. Given a fixed time-instance $t_0 \geq 0$, and a length $\tau > 0$ of the measurement horizon $[t_0, t_0+\tau]$, the *projected distribution* of the wealth from trading is usually calculated under the simplifying assumptions that

(1) the proportions of the wealth $\{\boldsymbol{\zeta}(t)\}_{t\in[t_0,t_0+\tau]}$ invested in various securities, as well as
(2) the market coefficients $\{\boldsymbol{\alpha}(t)\}_{t\in[t_0,t_0+\tau]}$ and $\{\boldsymbol{\sigma}(t)\}_{t\in[t_0,t_0+\tau]}$

will stay constant and equal to their present values throughout the time interval $[t_0, t_0+\tau]$. The wealth equations (2.3) and (2.4) yield that the **projected wealth loss** is - conditionally on $\mathcal{F}_{t_0}$ - distributed as $L = L(X(t_0), \zeta_{\boldsymbol{\mu}}(t_0), \zeta_{\boldsymbol{\sigma}}(t_0))$, where the law of $L(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ is the one of

$$x\Big(1 - \exp(Y(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}))\Big), \tag{2.16}$$

in which $Y(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ is a normal random variable with mean $\tilde{Q}(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})\tau$ and the standard deviation $\sqrt{\tau}\zeta_{\boldsymbol{\sigma}}$. The quantities $\zeta_{\boldsymbol{\mu}}(t_0)$ and $\zeta_{\boldsymbol{\sigma}}(t_0)$ are the portfolio rate of return and volatility, defined in (2.9).

Remark 2.11. The notion of the projected distribution of wealth as defined above has first appeared in the financial literature in [CHI07] in the context of constant coefficients. As one of the referees points out, while it is reasonable to keep the portfolio proportions constant throughout the measurement horizon $[t_0, t_0+\tau]$, the same cannot be said about the constancy of the market-coefficients $\boldsymbol{\alpha}(\cdot)$ and $\boldsymbol{\sigma}(\cdot)$. Indeed, under the conditions encountered in financial practice, the random evolution of $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ will typically lead to a more dispersed wealth distribution, and, consequently, to an under-estimate of the riskiness of the current position. The reason for such an assumption is the fact that it leads to a log-normally distributed wealth, which, in turn, greatly simplifies the analysis and leads to explicit form of the optimal policies. A simple and practically implementable way out of this predicament is to retain the assumption of the constancy of the market coefficients throughout the measurement horizon, but to use a "corrected" versions $\check{\boldsymbol{\sigma}}$ and $\check{\boldsymbol{\alpha}}$ of the current values $\boldsymbol{\sigma}(t_0)$ and $\boldsymbol{\alpha}(t_0)$ of the processes $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$. These, corrected, versions should correspond to a normal approximation of the true distribution of the investor's wealth at time $t_0+\tau$, and can be obtained in closed from in many of the models used in practice (see [SS91] for the case of stochastic volatility from Example 2.7 (2)). As the reader can easily check, such a corrected distribution will still lead to a portfolio-constraint compliant with Definition 2.9. In the case when processes driving the market coefficients are Markovian, a closed-form expression for the distribution of the wealth at time $t_0+\tau$ is available, and the conditions in the Definition 2.9 can be checked, one can use this exact distribution instead of the projected one. The authors have been unable, however, to identify any interesting cases where such a procedure is possible. Moreover, we feel that the approximation approach described above is more feasible for the practical application for yet another reason: even if one is able to identify the exact distribution of the wealth at time $t_0+\tau$, one still faces the much more difficult problem of estimation of the coefficients $\boldsymbol{\alpha}(t_0)$ and $\boldsymbol{\sigma}(t_0)$. We leave the implementation of a practical solution to this serious predicament for future research.

2.6.2. *Risk limits.* The purpose of this subsection is to define and expose certain properties of the risk measures (VaR, TVaR and LEL) discussed in the Introduction. Each one of these will be introduced through a family of random sets depending on the present values of the market coefficients, just like the ones in Definition 2.9. Put differently, our three risk measures will give rise to a random, wealth-dependent portfolio constraints. Strictly speaking, VaR, TVaR and LEL define *families* of risk measures, parameterized by exogenously chosen percentile parameter α, as well as the risk constraint parameters $a_V^{\text{abs}}, a_T^{\text{abs}}, a_L^{\text{abs}} > 0$ and $a_V^{\text{rel}}, a_T^{\text{rel}}, a_L^{\text{rel}} \in (0,1)$. We will assume that α is fixed and constant and that it satisfies $\alpha \in (0, 1/2)$. This technical assumption relates well to the practice where the typical values of $\alpha = 0.05$, or $\alpha = 0.1$ are used. It will be assumed through the rest of the paper that these parameters are arbitrarily chosen and fixed. Together with the market coefficients and the measurement horizon τ, they will play the role of "global variables".

Definition 2.12. The **value-at-risk** $\mathrm{VaR} = \mathrm{VaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ - corresponding to the current wealth x, the portfolio rate of return $\zeta_{\boldsymbol{\mu}}$ and volatility $\zeta_{\boldsymbol{\sigma}}$ - is the positive part of the upper α-percentile of the projected loss distribution $L = L(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$, i.e.,

$$\mathrm{VaR} = \gamma_\alpha^+ = \max(0, \gamma_\alpha), \text{ where } \gamma_\alpha \text{ uniquely satisfies } \mathbb{P}[L \geq \gamma_\alpha] = \alpha.$$

Definition 2.13. The **tail value-at-risk** $\mathrm{TVaR} = \mathrm{TVaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ is the positive part of the mean of the distribution of the projected loss distribution, conditioned to take a value above its upper α-percentile, i.e.,

$$\mathrm{TVaR} = w_\alpha^+, \text{ where } \gamma_\alpha \text{ satisfies } \mathbb{P}[L \geq \gamma_\alpha] = \alpha, \text{ and } w_\alpha = \mathbb{E}[L | L \geq \gamma_\alpha].$$

Our third measure of risk - LEL - is similar to TVaR, with one significant difference: it does not take the market rate-of-return in consideration. More precisely, we have the following definition

Definition 2.14. The **limited expected loss** $\mathrm{LEL} = \mathrm{LEL}(x, \zeta_{\boldsymbol{\sigma}})$ is the tail value-of-risk corresponding to the loss distribution $L = L(x, 0, \zeta_{\boldsymbol{\sigma}})$ in which the portfolio rate of return is set to 0.

Remark 2.15.

(1) In the common case when the financial market admits an equivalent martingale measure $\mathbb{Q}$, LEL can be interpreted as the TVaR calculated under $\mathbb{Q}$. The reader will easily convince him- or herself that, within our modelling framework at least, LEL will not depend on the choice of $\mathbb{Q}$, should there exist more than one.

(2) Definitions 2.12 and 2.13 differ slightly from the definitions of the Value at Risk and Tail Value at Risk given in [FS04a]: positive parts (not present in [FS04a]) are introduced in order to penalize only losses. Otherwise, it could happen that the induced constraints would, effectively, require the investor to make a certain, positive, return.

2.6.3. *Relative versions of risk measures.* All three VaR, TVaR and LEL measure the risk of a large loss in absolute terms. If we define the *relative projected wealth loss* as the distribution of the positive quantity $\frac{X^{\boldsymbol{\zeta}}(t_0) - X^{\boldsymbol{\zeta}}(t_0+\tau)}{X^{\boldsymbol{\zeta}}(t_0)}$ (under the simplifying assumptions 1. and 2. from paragraph 2.6.1 above), definitions of the analogous relative quantities VaR_r, TVaR_r and LEL_r can readily be given. In fact, due to the multiplicative structure of the wealth equations (2.3) and (2.4), we have the following expressions

$$\begin{gathered} \mathrm{VaR}_r(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) = \frac{\mathrm{VaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})}{x}, \qquad \mathrm{TVaR}_r(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) = \frac{\mathrm{TVaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})}{x}, \text{ and} \\ \mathrm{LEL}_r(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) = \frac{\mathrm{LEL}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})}{x}. \end{gathered} \tag{2.17}$$

As we would expect, the relative risk limits VaR_r, TVaR_r and LEL_r no longer depend on the current level of wealth x.

2.6.4. *Some explicit expressions.* Thanks to the fact that the distribution appearing in (2.16) is normal, explicit formulae can be given for the values of all three risk measures appearing above.

Proposition 2.16. *For $\zeta_{\boldsymbol{\mu}} \in \mathbb{R}$ and $\zeta_{\boldsymbol{\sigma}} > 0$, we have*

$$\mathrm{VaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) = x\left[1 - \exp\left(\tilde{Q}(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})\tau + N^{-1}(\alpha)\zeta_{\boldsymbol{\sigma}}\sqrt{\tau}\right)\right]^{+} \tag{2.18}$$

$$\mathrm{TVaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) = x\left[1 - \tfrac{1}{\alpha}e^{\tau(r+\zeta_{\boldsymbol{\mu}})}N(N^{-1}(\alpha) - \zeta_{\boldsymbol{\sigma}}\sqrt{\tau})\right]^{+}, \text{ and} \tag{2.19}$$

$$\mathrm{LEL}(x, \zeta_{\boldsymbol{\sigma}}) = x\left[1 - \tfrac{1}{\alpha}e^{r\tau}N\left(N^{-1}(\alpha) - \zeta_{\boldsymbol{\sigma}}\sqrt{\tau}\right)\right]^{+}, \tag{2.20}$$

where $N : \mathbb{R} \to (0, 1)$ is the cumulative distribution function of a standard normal random variable.

Proof. See Appendix A. □

2.6.5. *Constraints corresponding to risk measures.* For a constant $x > 0$, a vector $\boldsymbol{\mu} \in \mathbb{R}^n$ and a matrix $\boldsymbol{\sigma} \in \mathbb{R}^{n\times m}$, define

$$\tilde{F}_V^{\mathrm{abs}}(x, \boldsymbol{\mu}, \boldsymbol{\sigma}) = \left\{\boldsymbol{\zeta} \in \mathbb{R}^n \,:\, \mathrm{VaR}(x, \boldsymbol{\zeta}^T\boldsymbol{\mu}, ||\boldsymbol{\zeta}^T\boldsymbol{\sigma}||) \leq a_V^{\mathrm{abs}}\right\}, \tag{2.21}$$

where $a_V^{\mathrm{abs}} > 0$ is an exogenously defined constant. In words, $\tilde{F}_V^{\mathrm{abs}}$ is the set of all portfolio proportion vectors $\boldsymbol{\zeta} \in \mathbb{R}^n$ such that the loss incurred by keeping a fixed portfolio proportions $\boldsymbol{\zeta}$ in a market with constant rate-of-return $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, over a time horizon $[t, t+\tau]$ and with current wealth x results in no violation of the VaR risk limit. Thanks to Proposition 2.16, one can check that the correspondence $\tilde{F}_V^{\mathrm{abs}}$ from above is, in fact, a special case of a correspondence $F_{(f,h)}$ from Definition 2.9 (see Appendix A.)

Using the other 5 risk measures ($\mathrm{TVaR}, \mathrm{LEL}, \mathrm{VaR}_r, \mathrm{TVaR}_r$ and LEL_r) constraint sets $\tilde{F}_T^{\mathrm{abs}}$, $\tilde{F}_L^{\mathrm{abs}}$, $\tilde{F}_V^{\mathrm{rel}}$, $\tilde{F}_T^{\mathrm{rel}}$ and $\tilde{F}_L^{\mathrm{rel}}$, as well as their random-correspondence versions F_T^{abs}, F_L^{abs}, F_V^{rel}, F_T^{rel} and F_L^{rel} (constructed as in (2.15)) can be defined as in (2.21). One should note that for the relative versions the dependence on the current wealth level $X^{\boldsymbol{\zeta}}(t)$ is lost, and the portfolio constraint set will depend on the values of the market coefficients only. One can easily check that for these, relative, versions, functions f and h can be chosen so that the condition (B) in Definition 2.9 is satisfied. For the absolute versions, on the other hand, the condition (A) will be met.

2.7. **The optimization problem.** We finish the section by the formulation of our central problem. Given a choice of the constraint $\mathcal{A} = \mathcal{A}_{(f,h)}$ as in Definition 2.9, we are searching for a portfolio-proportion process $\boldsymbol{\zeta}^*(t) \in \mathcal{A}$ such that, for all $\boldsymbol{\zeta}(t) \in \mathcal{A}$,

$$\liminf_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} \geq \liminf_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}}(t))}{t}, \text{ a.s.}$$

Remark 2.17. While it is intimately related to the problem of maximizing logarithmic utility $\mathbb{E}[\log(X^{\boldsymbol{\zeta}}(T))]$, the ergodic problem we address here differs considerably from it - mainly in its dependence on the ergodicity of the market coefficients. The limiting nature of the objective criterion corresponds to a long-term average of the underlying controlled stochastic processes, while the classical logarithmic utility can only be applied to a finite (and fixed) time horizons. Moreover, the dependence of the constraint set on the current wealth (in the absolute case) rules out the naïve myopic approach characteristic of the behavior of a logarithmic investors on finite horizons. The relative case is much simpler and we can, in fact, treat both logarithmic and ergodic growth problems on the same footing (see the second part of the Main Theorem 3.1).

3. Main results

Our main result - Theorem 3.1 - summarizes the central findings of the manuscript. Its proof is the content of Section 4, below.

Theorem 3.1. *Let the financial market $\{S_0(t), S_1(t), \dots, S_n(t)\}_{t\in[0,\infty)}$ be defined as in* (2.1)*, with the coefficients $r > 0$, $\{\boldsymbol{\alpha}(t)\}_{t\in[0,\infty)}$ and $\{\boldsymbol{\sigma}(t)\}_{t\in[0,\infty)}$ satisfying Assumptions 2.1, 2.4 and 2.5. Furthermore, let the functions f and h, as well as the corresponding admissible class $\mathcal{A} = \mathcal{A}_{(f,h)}$ be as in Definition 2.9. Then the following statements hold.*

(1) **Absolute constraints**

Suppose that the function h satisfies the assumption set (A) from Definition 2.9. Let $\{\boldsymbol{\zeta}_M(t)\}_{t\in[0,\infty)}$ be the Merton-proportion *process defined in* (2.10)*. There exists a stochastic processes $\{\beta^*(t)\}_{t\in[0,\infty)}$ and $\{\beta^\infty(t)\}_{t\in[0,\infty)}$ taking values in $(0, 1]$ such that the vector-valued processes $\{\boldsymbol{\zeta}^*(t)\}_{t\in[0,\infty)}$ and $\{\boldsymbol{\zeta}^\infty(t)\}_{t\in[0,\infty)}$, defined by*

$$\boldsymbol{\zeta}^*(t) = \beta^*(t)\boldsymbol{\zeta}_M(t),\ \boldsymbol{\zeta}^\infty(t) = \beta^\infty(t)\boldsymbol{\zeta}_M(t), \tag{3.1}$$

have the following properties

(a) *both $\{\boldsymbol{\zeta}^*(t)\}_{t\in[0,\infty)}$ and $\{\boldsymbol{\zeta}^\infty(t)\}_{t\in[0,\infty)}$ are cáglád and define strictly positive wealth processes*

$$X^{\boldsymbol{\zeta}^*}(t) = X(0)\exp\Big(\int_0^t Q(t, \boldsymbol{\zeta}^*(t))\,dt + \int_0^t (\boldsymbol{\zeta}^*(t))^T\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\Big),$$
$$X^{\boldsymbol{\zeta}^\infty}(t) = X(0)\exp\Big(\int_0^t Q(t, \boldsymbol{\zeta}^\infty(t))\,dt + \int_0^t (\boldsymbol{\zeta}^\infty(t))^T\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\Big),$$

(b) *$\boldsymbol{\zeta}^\infty(t)$ is the unique projection of $\boldsymbol{\zeta}_M(t)$ onto the limiting constraint set*

$$F(t,\infty) = F_{(f,h)}(t,\infty) = \cap_{x>0}F(t,x),$$

under the metric $d_{\boldsymbol{\sigma}(t)}$ on $\mathbb{R}^n$ defined by (2.11).

(c) $\boldsymbol{\zeta}^*(t)$ *is the unique* $d_{\boldsymbol{\sigma}(t)}$*-projection of* $\boldsymbol{\zeta}_M(t)$ *onto the constraint set* $F(t, X^{\boldsymbol{\zeta}^*}(t))$.

(d) $\{\boldsymbol{\zeta}^*(t)\}_{t\in[0,\infty)}$ *and* $\{\boldsymbol{\zeta}^\infty(t)\}_{t\in[0,\infty)}$ *are* (f,h)*-admissible and*

$$\lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} = \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^\infty}(t))}{t} = r + Z(x^2\delta(x)),$$

where $Z(\cdot)$ *is the random variable introduced in Assumption 2.5, and* $\delta : [0,\infty) \to (0,1]$ *is a non-negative continuous function depending only on the constraint type, but independent of the market coefficients.*

(e) *More precisely, for* $\lambda \geq 0$, $\delta(\lambda) = \min(g(\lambda), 1)$, *where* $g(\lambda)$ *is the unique positive solution of the equation*

$$f(g(\lambda)\lambda^2, g(\lambda)\lambda) = 0. \tag{3.2}$$

Additionally, with $\delta^*(\lambda, x) = \min(g^*(\lambda, x), 1)$, *where* $g^*(\lambda, x)$ *is is the unique positive solution of the equation*

$$f(g^*(\lambda, x)\lambda^2, g^*(\lambda, x)\lambda) = h(x), \;\; x, \lambda > 0, \tag{3.3}$$

we have

$$\boldsymbol{\zeta}^*(t) = \delta^*(||\boldsymbol{\zeta}_M(t)\boldsymbol{\sigma}(t)||, X^{\boldsymbol{\zeta}^*}(t)) \text{ and } \boldsymbol{\zeta}^\infty(t) = \delta(||\boldsymbol{\zeta}_M(t)\boldsymbol{\sigma}(t)||). \tag{3.4}$$

(f) *Both* $\boldsymbol{\zeta}^*(t)$ *and* $\boldsymbol{\zeta}^\infty(t)$ *are growth optimal in the sense that*

$$\liminf_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}}(t))}{t} \leq \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} = \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^\infty}(t))}{t}, \;\; a.s.,$$

for any $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)} \in \mathcal{A}_{(f,h)}$.

(2) **Relative constraints**

Suppose that the function h *satisfies the assumption (B) from Definition 2.9. Define the process* $\{\boldsymbol{\zeta}^r(t)\}_{t\in[0,\infty)}$ *as a projection of the Merton proportion* $\boldsymbol{\zeta}_M(t)$ *onto the (wealth-independent) constraint set* $F(t)$, *under the metric* $d_{\boldsymbol{\sigma}}$. *Then* $\boldsymbol{\zeta}^r(t)$ *is both* log- *and growth-optimal in the class* $\mathcal{A}_{(f,h)}$, *i.e.,*

$$\liminf_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}}(t))}{t} \leq \liminf_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^r}(t))}{t}, \;\; a.s.,$$

and

$$\mathbb{E}[\log(X^{\boldsymbol{\zeta}^r}(\bar{\tau})) - \log(X^{\boldsymbol{\zeta}}(\bar{\tau}))] \leq 0,$$

for all $\boldsymbol{\zeta} \in \mathcal{A}_{(f,h)}$, *and all* $[0,\infty)$*-valued stopping times* $\bar{\tau}$, *interpreted as time-horizons.*

Remark 3.2. The central message of the main Theorem 3.1 is the following: even though the absolute constraints mix the wealth dependence and the risk-constraints in a complicated way, it turns out that the problem still admits a simple solution - just project the unconstrained optimal portfolio proportion onto the constraint set. Moreover, our analysis shows that in the conjunction with the ergodic criterion, the absolute wealth constraints are (eventually) so strong that the agent is forced to invest in a severely restricted way. In the case of a VaR-constraint, for example, no loss whatsoever is tolerated (in the asymptotic sense). On the other hand, we provide another optimal policy $\boldsymbol{\zeta}^*$ which performs much better on finite horizons, but attains the same asymptotic growth. Finally, we provide an explicit formula for the optimal asymptotic growth which depends in a simple way on the primitives of the model.

In the relative case, things are much simpler, and we show that asymptotic optimality is equivalent to finite-horizon optimality for any choice of the horizon. The results obtained generalize directly the related results in [CHI07].

3.1. **Some explicit examples.** Before we present the proof of Theorem 3.1 in the following section, we illustrate some of its features through an example where the optimal asymptotic growth-rates can be computed explicitly.

Example 3.3.

(1) **Constant coefficients.** Suppose that the coefficients $\boldsymbol{\mu}(t) \equiv \boldsymbol{\mu}$ and $\boldsymbol{\sigma}(t) \equiv \boldsymbol{\sigma}$ are constant. In that case the ergodic Assumption 2.5 is trivially satisfied, $Z(\varphi)$ is a constant random variable for each φ, and and we have $Z(\varphi) = \varphi(||\boldsymbol{\zeta}_M^T \boldsymbol{\sigma}||)$. Therefore,
$$\lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} = \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^\infty}(t))}{t} = r + ||\boldsymbol{\zeta}_M^T \boldsymbol{\sigma}||^2 \delta(||\boldsymbol{\zeta}_M^T \boldsymbol{\sigma}||).$$
In the case where the constraints are such that $\boldsymbol{\zeta}_M \in F(t,x)$ for all t, x, we clearly have $\delta(x) = 1$, for all x and we recover the well known Merton's solution to the growth-rate optimization problem. In the case of VaR-, TVar- and LEL-constraints, the explicit expression for δ (and, thus, for β^* and β^∞) can be obtained from the explicit expressions in Proposition 2.16 and the representation (3.2) from Theorem 3.1. While elementary, these calculations are quite tedious and their results are not very illuminating, so we omit them.

(2) **Periodic coefficients.** In this case, the coefficient processes $\boldsymbol{\mu}(t)$ and $\boldsymbol{\sigma}(t)$ are assumed to be deterministic and periodic with period T_0. It is not hard to see that the Assumption 2.5 is still satisfied and that we have $Z(\varphi) = \frac{1}{T_0}\int_0^{T_0} \varphi(||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||)\,dt$, so that
$$\lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} = \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^\infty}(t))}{t} = r + \frac{1}{T_0}\int_0^{T_0} ||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||^2 \delta(||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||)\,dt.$$

(3) **Stochastic volatility.** While the calculations with the realistic constraints like VaR, TVar and LEL are possible, but quite messy in the stochastic volatility model as presented in

Example 2.7 (2), the unconstrained case can be treated with ease. Indeed, then $\delta(x) = 1$, for all x and, using the discussion in Example 2.7 (2), we have

$$\lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} = \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^\infty}(t))}{t} = r + \frac{\mu^2}{\bar{\sigma}^2},$$

where

$$\bar{\sigma} = \left(\int_{\mathbb{R}} \frac{\sqrt{\nu}}{\sqrt{\pi}} \Sigma(x)^{-2} e^{-\nu(x-\bar{V})^2}\, dx \right)^{-2}.$$

In words, the optimal growth-rate in the stochastic volatility market matches the optimal growth rate in a constant-coefficient market in which the volatility is a harmonic-type mean of the stochastic volatility $\Sigma(\cdot)$ over the invariant measure.

4. Analysis

From this point onward, we fix a pair of functions (f, h) as in Definition 2.9, and drop all related subscripts from the notation. If a distinction between the relative and the absolute case is needed, it will be made explicit, and the unified notation $F(t, x)$ will be used instead of $F(t)$ for the relative constraints. Unless stated otherwise, statements and definitions made for the values of random processes, fields and correspondences are assumed to hold for all $t \in [0, \infty)$, a.s.

4.1. Properties of the constraint sets.

Several analytical properties of the (instantaneous) constraint sets $F(t, x)$ are established in this Subsection.

Lemma 4.1. *For a vector $\boldsymbol{\mu} \in \mathbb{R}^n$ and a full-rank matrix $\boldsymbol{\sigma} \in \mathbb{R}^{n\times m}$, let $\boldsymbol{\zeta}_M = (\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1}\boldsymbol{\mu}$. Then the following inequality holds for each $\boldsymbol{\zeta} \in \mathbb{R}^n$*

$$\boldsymbol{\zeta}^T\boldsymbol{\mu} \le ||\boldsymbol{\zeta}^T\boldsymbol{\sigma}||\, ||\boldsymbol{\zeta}_M^T\boldsymbol{\sigma}||.$$

Proof. Since $\boldsymbol{\mu} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\zeta}_M$, we have

$$\boldsymbol{\zeta}^T\boldsymbol{\mu} = (\boldsymbol{\sigma}^T\boldsymbol{\zeta})^T(\boldsymbol{\sigma}^T\boldsymbol{\zeta}_M) \le ||\boldsymbol{\zeta}^T\boldsymbol{\sigma}||\, ||\boldsymbol{\zeta}_M^T\boldsymbol{\sigma}||,$$

by the Cauchy-Buniakowski-Schwarz inequality. □

The following lemma gives an upper bound on the size of the constraint sets.

Lemma 4.2. *There exist constants $C_i > 0$, $i = 1, 2, 3$ (independent of the market coefficients) such that*

$$||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)|| \le C_1||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)|| + C_2\sqrt{h(x) + C_3},$$

whenever $\boldsymbol{\zeta} \in F(t, x)$.

Consequently, each $F(t,x)$ is contained in a $d_{\boldsymbol{\sigma}(t)}$-ball of (possibly infinite) radius $C_1||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||+C_2\sqrt{h(x)+C_3}$ around the origin.

Proof. Without loss of generality we assume that $h(x) < \infty$. Lemma 4.1 in conjunction with property (3) from Definition 2.9 yields that, for each $\boldsymbol{\zeta} \in F(t,x)$, we have

$$\begin{aligned} 0 &\geq f(\boldsymbol{\zeta}^T\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)|| \geq \kappa_1||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)||^2 - \kappa_2\boldsymbol{\zeta}^T\boldsymbol{\mu}(t) - \kappa_3 - h(x) \\ &\geq \kappa_1||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)||^2 - \kappa_2||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)|| - \kappa_3 - h(x), \end{aligned}$$

for some constants $\kappa_i > 0$, $i = 1,2,3$. Consequently, a simple estimate based on the quadratic inequality for $||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)||$ above and the fact that $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for $a,b \geq 0$ yields

$$||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)|| \leq C_1||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}|| + C_2\sqrt{h(x)+C_3}, \text{ where } k_1 = \frac{\kappa_2}{\kappa_1}, C_2 = \sqrt{\frac{\kappa_3}{\kappa_1}}, C_3 = \kappa_3.$$

□

The following Proposition is a simple corollary of Lemma 4.2 above is

Proposition 4.3. *In the relative case, the constraint set $F(t)$ is convex and compact. In the absolute case, $F(t,x)$ is always convex and either compact or equal to the whole $\mathbb{R}^n$, depending on whether $x > x_0$ or $x \leq x_0$.*

Proof. It is clear that $F = F(t,x)$ equals the whole of $\mathbb{R}^n$ when $h(x) = +\infty$. We can suppose, therefore, that $h(x) \in \mathbb{R}$, treat both absolute and relative cases together, and establish compactness and convexity.

Convexity is inherited directly from the joint convexity of the function $(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \mapsto f(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$, its increase in the second variable, and the convexity of the mappings $\boldsymbol{\zeta} \mapsto \boldsymbol{\zeta}^T\mu$ and $\boldsymbol{\zeta} \mapsto ||\boldsymbol{\zeta}^T\boldsymbol{\sigma}||$.

To establish compactness, we turn to Lemma 4.2 and conclude that $F(t,x)$ is a bounded set, since the metric $d_{\boldsymbol{\sigma}(t)}$ and the Euclidean metric d are equivalent. Finally, closedness of $F(t,x)$ follows from joint continuity of the function f. □

4.2. **Structure of the projections on the constraint sets.** Proposition 4.5 below exposes an interesting property of the $d_{\boldsymbol{\sigma}(t)}$-projection of the Merton-proportion process $\boldsymbol{\zeta}_M(t)$ onto the constraint set $F(t,x)$ - namely, that it is collinear with $\mathbf{0}$ and $\boldsymbol{\zeta}_M(t)$ and lies between them. This unexpected property of the constraints is going to be instrumental for the arguments in the sequel. In preparation for the proof of Proposition 4.5, we need to introduce a random field $g : \Omega \times [0,T] \times [0,\infty) \to \mathbb{R}$ and identify some of its properties; for $\beta \in [0,\infty)$ we set

$$g(t,\beta) = f(\beta||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||^2, \beta||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||), \ t \in [0,\infty). \tag{4.1}$$

Lemma 4.4. *The following hold true for the random field g, defined in* (4.1).

(1) *for every $(\omega, t) \in \Omega \times [0, \infty)$, $g(t, \cdot)$ is a convex, continuously differentiable function.*

(2) *for every $(\omega, t) \in \Omega \times [0, \infty)$, $g(t, 0) = g(0, 0) < 0$, and*

(3) *for every $\beta > 0$, $T > 0$, $\sup_{t \in [0,T]} |g(t, \beta)| < \infty$, a.s.*

(4) *for every $(\omega, t) \in \Omega \times [0, \infty)$, and every $c > 0$, the equation $g(t, \beta) = c$ has a unique solution.*

Proof. Property (1) follows from the joint convexity of f, (2) is a restatement of the fact that $f(0, 0) < 0$, and (3) is a consequence of continuity of f, coupled with the local boundedness of the market coefficients. To establish (4), we recall that $g(t, \cdot)$ is convex, $g(t, 0) < 0$ and $\lim_{\beta \to \infty} g(t, \beta) = +\infty$, thanks to the equation (2.14) in Definition 2.9. $\square$

Proposition 4.5. *Choose $x \in (0, \infty]$, and let $\pi_F(\boldsymbol{\zeta}_M(t))$ denote the projection of the Merton-proportion $\boldsymbol{\zeta}_M(t)$ process onto the convex set $F(t, x)$, with respect to the metric $d_{\boldsymbol{\sigma}(t)}$. Then there exists a constant $\beta(t, x)$ - defining a random field $\beta : \Omega \times [0, T] \times [0, \infty) \to (0, 1]$ - such that*

$$\pi_F(\boldsymbol{\zeta}_M(t)) = \beta(t, x)\boldsymbol{\zeta}_M(t).$$

Moreover, $\beta(t, x) = 1$ when $h(x) = +\infty$. Otherwise, $\beta(t, x) = 1 \wedge b(t, x)$, where $b(t, x)$ uniquely satisfies

$$g(t, b(t, x)) = h(x).$$

Proof. Existence and uniqueness of the projection $\pi_F(\boldsymbol{\zeta}_M(t))$ are consequences of the the compactness of the set $F(t, x)$ and strict convexity of the norm $d_{\boldsymbol{\sigma}(t)}$. All statements of the proposition are trivial if $\boldsymbol{\zeta}_M(t) \in F(t, x)$, so we can freely assume that $\boldsymbol{\zeta}_M(t) \notin F(t, x)$. In particular, this assumption forces $h(x) < \infty$.

The mapping $\boldsymbol{\zeta} \mapsto \boldsymbol{\sigma}(t)^T \boldsymbol{\zeta}$ from $\mathbb{R}^n$ to $\text{Range}(\boldsymbol{\sigma}(t)^T) \subseteq \mathbb{R}^m$ is a linear isomorphism. Moreover, it is also an isometry when $\mathbb{R}^n$ is equipped with the metric $d_{\boldsymbol{\sigma}(t)}$ and $\text{Range}(\boldsymbol{\sigma}(t)^T)$ with the standard Euclidean metric d. Therefore, the image $\boldsymbol{\sigma}(t)^T \pi_F(\boldsymbol{\zeta}_M(t))$ of the projection $\pi_F(\boldsymbol{\zeta}_M(t))$ is the Euclidean projection $\pi_{F'}(\boldsymbol{\rho}_M(t))$ of the image $\boldsymbol{\rho}_M(t) = \boldsymbol{\sigma}(t)^T \boldsymbol{\zeta}_M(t)$, onto the image $F'(t, x) = \boldsymbol{\sigma}(t) F(t, x)$ of the constraint set $F(t, x)$. Consequently, it will be enough to show that $\pi_{F'}(\boldsymbol{\rho}_M(t))$ is of the form $\beta(t, x)\boldsymbol{\rho}_M(t)$, and that the mapping β has the desired properties.

With $||\boldsymbol{\rho}||_{\boldsymbol{\sigma}(t)}$ defined as $d_{\boldsymbol{\sigma}(t)}(\boldsymbol{\rho}, \mathbf{0})$, for $\boldsymbol{\rho} \in \text{Range}(\boldsymbol{\sigma}(t)^T)$, we have

$$F(t, x) = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^n \,:\, f(\boldsymbol{\zeta}^T \boldsymbol{\mu}(t), ||\boldsymbol{\zeta}^T \boldsymbol{\sigma}(t)||) \le h(x) \right\}, \text{ and}$$
$$F'(t, x) = \left\{ \boldsymbol{\rho} \in \text{Range}(\boldsymbol{\sigma}(t)^T) \,:\, f(\boldsymbol{\rho}^T \boldsymbol{\rho}_M(t), ||\boldsymbol{\rho}||_{\boldsymbol{\sigma}(t)}) \le h(x) \right\}.$$

The equality $\boldsymbol{\zeta}_M(t)\boldsymbol{\mu}(t) = ||\boldsymbol{\zeta}_M(t)\boldsymbol{\sigma}(t)||^2$ and Lemma 4.4 imply that there exists unique number $\beta(t,x) \in (0,1]$ such that

$$\begin{cases} \beta\boldsymbol{\rho}_M(t) \in F'(t,x), & \text{for } \beta \in [0, \beta(t,x)], \text{ and} \\ \beta\boldsymbol{\rho}_M(t) \notin F'(t,x), & \text{for } \beta \in (\beta(t,x), 1]. \end{cases} \tag{4.2}$$

Moreover, remembering the assumption $\boldsymbol{\zeta}_M(t) \notin F(t,x)$ (or, equivalently $\boldsymbol{\rho}_M(t) \notin F'(t,x)$), we can easily see that $\beta(t,x) < 1$ must be of the form $\beta(t,x) = g(t,x)$ (where g is defined in (4.1) above).

It is our goal to show that $\boldsymbol{\rho}_0(t) \triangleq \beta(t,x)\boldsymbol{\rho}_M(t)$ coincides with the projection $\pi_{F'}(\boldsymbol{\rho}_M(t))$. To progress with this claim, let P denote the semi-space

$$P = \left\{\boldsymbol{\rho} \in \mathrm{Range}(\boldsymbol{\sigma}(t)^T) \,:\, (\boldsymbol{\rho} - \boldsymbol{\rho}_0(t))^T \boldsymbol{\rho}_M(t) > 0\right\},$$

supported by a hyperplane through $\boldsymbol{\rho}_0(t)$, perpendicular to $\boldsymbol{\rho}_M(t)$. Thanks to the assumption $\boldsymbol{\rho}_M(t) \notin F'(t,x)$, the vector $\boldsymbol{\rho}_M(t)$ cannot be equal to $\mathbf{0}$, and so P does not degenerate to the whole $\mathrm{Range}(\boldsymbol{\sigma}(t)^T)$. The points in $P^c \setminus \{\boldsymbol{\rho}_0(t)\}$ are further away from $\boldsymbol{\rho}_M(t)$ than $\boldsymbol{\rho}_0(t)$ is, so it will be enough to show that $P \cap F'(t,x) = \emptyset$. Suppose, to the contrary, that there exists a vector $\bar{\boldsymbol{\rho}} \in P$ such that $\bar{\boldsymbol{\rho}} \in F'(t,x)$, i.e., $f(\bar{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t), ||\bar{\boldsymbol{\rho}}||_{\boldsymbol{\sigma}(t)}) \leq h(x)$. Let

$$\hat{\boldsymbol{\rho}} = ||\bar{\boldsymbol{\rho}}||_{\boldsymbol{\sigma}(t)} \frac{\boldsymbol{\rho}_M(t)}{||\boldsymbol{\rho}_M(t)||_{\boldsymbol{\sigma}(t)}}.$$

Since $\hat{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t) = ||\bar{\boldsymbol{\rho}}^T||_{\boldsymbol{\sigma}(t)} ||\boldsymbol{\rho}_M(t)||_{\boldsymbol{\sigma}(t)} \geq \bar{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t)$, and since the function f is decreasing in its first variable, we have

$$f(\hat{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t), ||\hat{\boldsymbol{\rho}}^T||) = f(\hat{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t), ||\bar{\boldsymbol{\rho}}||) \leq f(\bar{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t), ||\bar{\boldsymbol{\rho}}||) \leq h(x)$$

so $\hat{\boldsymbol{\rho}} \in F'(t,x)$. All three points $\mathbf{0}$, $\hat{\boldsymbol{\rho}}$ and $\boldsymbol{\rho}_0(t)$ are non-negative multiples of $\boldsymbol{\rho}_M(t)$, with $\boldsymbol{\rho}_0(t)$ being between the other two. Therefore, there exists a constant $\lambda \in [0,1]$ such that $\boldsymbol{\rho}_0(t) = \lambda\mathbf{0} + (1-\lambda)\hat{\boldsymbol{\rho}}$. Because $\hat{\boldsymbol{\rho}} \in P$, $\lambda > 0$. Thanks to the joint convexity of the function f, we have

$$\begin{aligned} f(\boldsymbol{\rho}_0(t)^T \boldsymbol{\rho}_M(t), ||\boldsymbol{\rho}_0(t)||) &= f((1-\lambda)\hat{\boldsymbol{\rho}}_0^T \boldsymbol{\rho}_M(t), ||(1-\lambda)\hat{\boldsymbol{\rho}}||) \\ &= f(\lambda\mathbf{0} + (1-\lambda)\hat{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t), \lambda||\mathbf{0}|| + (1-\lambda)||\hat{\boldsymbol{\rho}}||) \\ &\leq \lambda f(0,0) + (1-\lambda) f(\hat{\boldsymbol{\rho}}^T \boldsymbol{\rho}_M(t), ||\hat{\boldsymbol{\rho}}||) < h(x). \end{aligned}$$

Continuity of the mapping $\kappa \mapsto f(\kappa\boldsymbol{\rho}_M(t)^T \boldsymbol{\rho}_M(t), ||\kappa\boldsymbol{\rho}_M(t)_0||)$ (from Lemma 4.4) implies that there exists an open interval $(\underline{\kappa}, \overline{\kappa})$ around $\beta(t,x)$ such that $\kappa\boldsymbol{\rho}_M(t) \in F'$ for all $\kappa \in (\underline{\beta}, \overline{\beta})$. This is, however, in contradiction with (4.2). □

Remark 4.6. The proof of Proposition 4.5 above can be given without a recourse to the change-of-variable transformation $\boldsymbol{\rho} = \boldsymbol{\sigma}(t)^T \boldsymbol{\zeta}(t)$. We do this in order to help the reader's intuition by placing him or her in the familiar isotropic Euclidean setting.

The following Lemma plays a central role in the proof of Lemma 4.8 below. It establishes a uniform version of the Lipschitz property for the mapping $\beta(\cdot, \exp(\cdot))$.

Lemma 4.7. *There exists an increasing process $L : [0,\infty) \to (0,\infty)$ such that*

$$|\beta(t, e^{y_1}) - \beta(t, e^{y_2})| \le L(T)|y_2 - y_1| \ , \forall\, t \in [0,T],\ y_1, y_2 \in \mathbb{R}, \tag{4.3}$$

for each $T > 0$.

Proof. We fix a time horizon $T > 0$, a time instance $t \in [0,T]$, and a typical $\omega \in \Omega$. Without loss of generality we may assume that $x_1 = \exp(y_1) < x_2 = \exp(y_2)$ and that $\beta(t, x_2) < 1$, which, in turn, implies that $h(x_2) < \infty$. When $\beta(t, x_1) = 1$ then $g(t, 1) \le h(x_1)$, and we can find a unique $\bar{x}_1 > x_0$ with the property that $g(t, 1) = h(\bar{x}_1)$. Clearly $x_2 > \bar{x}_1 \ge x_1$. When $\beta(t, x_1) < 1$, we simply set $\bar{x}_1 = x_1$. In either case we have

$$h(\bar{x}_1) - h(x_2) = g(t, \beta(t, \bar{x}_1)) - g(t, \beta(t, x_2)) = g(t, \beta(t, x_1)) - g(t, \beta(t, x_2)). \tag{4.4}$$

Thanks to (4.4) and continuous differentiability and convexity of the function $g(t, \cdot)$ (see Lemma 4.4), we have

$$\begin{aligned} h(\bar{x}_1) - h(x_2) &= \int_{\beta(t,x_2)}^{\beta(t,x_1)} \frac{\partial}{\partial \beta} g(t, \xi)\, d\xi \ge (\beta(t, x_2) - \beta(t, x_1)) \frac{\partial}{\partial \beta} g(t, \beta(t, x_2)) \\ &\ge (\beta(t, x_2) - \beta(t, x_2)) \frac{g(t, \beta(t, x_2)) - g(0,0)}{\beta(t, x_2)} \\ &\ge (\beta(t, x_2) - \beta(t, x_1)) \frac{-g(0,0)}{\beta(t, x_2)}, \end{aligned} \tag{4.5}$$

where the last two inequalities follow from the convexity of $g(t, \cdot)$ and the fact that $g(t, 0) = g(0, 0)$, for any t. On the other hand, due to the convexity of $\tilde{h}(\cdot) = h(\exp(\cdot))$, we have

$$h(\bar{x}_1) - h(x_2) \le (y_2 - \log(\bar{x}_1))(-\tilde{h}'(\log(\bar{x}_1))) \le (y_2 - y_1)(-\tilde{h}'(y_T)), \tag{4.6}$$

where $y_T = \log(h^{-1}(\sup_{t\in[0,T]} g(t, 1))) > \log(x_0)$. Finally, as $\beta(t, x_2) < 1$, (4.5) and (4.6) can be combined to imply

$$|\beta(t, e^{y_2}) - \beta(t, e^{y_1})| = \beta(t, e^{y_1}) - \beta(t, e^{y_2}) \le L(T)(y_2 - y_1) = L(T)\, |y_2 - y_1|\,,$$

where $L(T) = \tilde{h}'(y_T)/g(0,0)$. $\square$

4.3. **Candidate optimal portfolio proportions.**

Lemma 4.8. *The following stochastic differential equation*

$$\begin{cases} dX^{\boldsymbol{\zeta}^*}(t) = X^{\boldsymbol{\zeta}^*}(t)\big[\big(r + (\boldsymbol{\zeta}^*)^T(t)\boldsymbol{\mu}(t)\big)\,dt + (\boldsymbol{\zeta}^*)^T(t)\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\big], \\ \qquad\qquad \text{where } \boldsymbol{\zeta}^*(t) = \beta(t, X^{\boldsymbol{\zeta}^*}(t))\boldsymbol{\zeta}_M(t), \\ X^{\boldsymbol{\zeta}^*}(0) = X(0) \end{cases} \tag{4.7}$$

has a unique strong solution in $[0,\infty)$.

Proof. It will be enough to choose a fixed, but arbitrary time horizon $[0,T]$, and prove existence and uniqueness of the solution $Y(t) = \log X^{\boldsymbol{\zeta}^*}(t)$ of the stochastic differential equation

$$\begin{cases} dY(t) = Q(t, \boldsymbol{\zeta}^*(t))\,dt + (\boldsymbol{\zeta}^*)^T(t)\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t) \\ \boldsymbol{\zeta}^*(t) = \beta(t, e^{Y(t)})\boldsymbol{\zeta}_M(t). \end{cases} \tag{4.8}$$

According to [Pro04, Theorem 7., p. 194] it will be enough to establish the Lipschitz property of the (cáglád) coefficients of (4.8), for each ω, uniformly in $t \in [0,T]$. In that direction, we note that the coefficient $(\boldsymbol{\zeta}^*)^T(t)\boldsymbol{\sigma}(t)$ of $d\boldsymbol{W}(t)$ satisfies the mentioned Lipschitz property thanks to Lemma 4.7 and local boundedness of $\boldsymbol{\sigma}(t)$. As for the dt-coefficient $Q(t, \boldsymbol{\zeta}^*(t))$, we only need to observe that

$$\begin{aligned} &|Q(t, \beta(t, e^{y_2}))\boldsymbol{\zeta}_M(t) - Q(t, \beta(t, e^{y_1}))\boldsymbol{\zeta}_M(t)| \\ &= \quad \left|\beta(t, e^{y^1}) - \beta(t, e^{y_1})\right| \left|1 - \frac{1}{2}(\beta(t, e^{y^1}) + \beta(t, e^{y_1}))\right| \left|\boldsymbol{\zeta}_M^T(t)\boldsymbol{\mu}(t)\right|, \end{aligned} \tag{4.9}$$

and use Lemma 4.7 and local boundedness of the process $\left|\boldsymbol{\zeta}_M^T(t)\boldsymbol{\mu}(t)\right|$. ☐

We introduce the process $\{\boldsymbol{\zeta}^\infty(t)\}_{t\in[0,\infty)}$, given by $\boldsymbol{\zeta}^\infty(t) = \beta(t,\infty)\boldsymbol{\zeta}_M^T(t)$ where $\beta(t,\infty) = \lim_{x\to\infty}\beta(t,x)$. It is readily seen that $\boldsymbol{\zeta}^\infty(t)$ is the $d_{\boldsymbol{\sigma}(t)}$-projection of $\boldsymbol{\zeta}_M(t)$ onto the limiting constraint set $F(t,\infty)$. Thanks to the previous Lemma, the process $\{\boldsymbol{\zeta}^*(t)\}_{t\in[0,\infty)}$ is uniquely determined by (4.7).

Corollary 4.9. $\{\boldsymbol{\zeta}^*(t)\}_{t\in[0,\infty)}, \{\boldsymbol{\zeta}^\infty(t)\}_{t\in[0,\infty)} \in \mathcal{A}$.

4.4. **The question of transience.** Before engaging in the proof of optimality of $\boldsymbol{\zeta}^*(t)$ and $\boldsymbol{\zeta}^\infty(t)$, we need to understand better the transience properties of the wealth process $X^{\boldsymbol{\zeta}}(t)$ for arbitrary $\boldsymbol{\zeta} \in \mathcal{A}$.

Lemma 4.10. *For* $\boldsymbol{\zeta}(t) \in \mathcal{A}$, *let* $\{X^{\boldsymbol{\zeta}}(t)\}_{t\in[0,\infty)}$ *be the corresponding wealth process. Then*

$$\lim_{t\to\infty}\frac{M^{\boldsymbol{\zeta}}(t)}{t} = 0, \;\; \text{on } \left\{\lim_{t\to\infty} X^{\boldsymbol{\zeta}}(t) = \infty\right\} \in \mathcal{F}_\infty,$$

where $M^{\boldsymbol{\zeta}}(t)$ *is the local martingale defined in* (2.8).

Proof. Let $A(t) = [M^{\boldsymbol{\zeta}}(t), M^{\boldsymbol{\zeta}}(t)]$ be the quadratic variation of $M^{\boldsymbol{\zeta}}(t)$. By Lemma 4.2 and Definition 2.9, there exists constants $D_i > 0$, $i - 1, 2, 3$ such that

$$A(t) \leq D_1 t + D_2 \int_0^t ||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||^2 \, du + D_3 \int_0^t h(X^{\boldsymbol{\zeta}}(u)) \, du. \tag{4.10}$$

Of course, the estimate above is only useful for (t, ω) where $h(X^{\boldsymbol{\zeta}}(t)) < \infty$. Fortunately, for each $\omega \in \mathrm{Tr}_X^{\boldsymbol{\zeta}} = \left\{\lim_{t\to\infty} X^{\boldsymbol{\zeta}}(t) = \infty\right\} \in \mathcal{F}_\infty$ there exists $T'(\omega) > 0$ such that $h(X^{\boldsymbol{\zeta}}(t)) < 1$ for all $t > T'(\omega)$. This is a direct consequence of the definition of the set $\mathrm{Tr}_X^{\boldsymbol{\zeta}}$ and the properties of the function h. Thus, the inequality (4.10) can be transformed into

$$A(t) \leq A(T'(\omega)) + D_1(t - T'(\omega)) + D_2 \int_{T'(\omega)}^t ||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||^2 \, du + D_3(t - T'(\omega)) \tag{4.11}$$

for $t \geq T'(\omega)$ on $\mathrm{Tr}_X^{\boldsymbol{\zeta}}$. Assumption 2.5 now implies that

$$\xi_0 \triangleq \limsup_{t\to\infty} \frac{A(t)}{t} \leq D_1 + D_2 Z(x^2) + D_3 < \infty, \text{ a.s. on } \mathrm{Tr}_X^{\boldsymbol{\zeta}},$$

where the operator Z is described in (2.12).

By the Theorem of Dambis, Dubins and Schwarz (see Theorem 4.6, p. 174 in [KS91]), there exists a Brownian motion $\{B_t\}_{t\in[0,\infty)}$ (possibly defined on the extended probability space) such that $M_t^{\boldsymbol{\zeta}} = B_{A(t)}$. By the Law of Large Numbers for Brownian motion (see Problem 9.3, p. 104 in [KS91]) we have (with the convention $\frac{0}{0} = 0$)

$$\lim_{t\to\infty} \frac{M^{\boldsymbol{\zeta}}(t)}{A(t)} = 0, \text{ on } \mathrm{Tr}_A^{\boldsymbol{\zeta}} \triangleq \left\{\lim_{t\to\infty} A(t) = +\infty\right\}.$$

On $(\mathrm{Tr}_A^{\boldsymbol{\zeta}})^c = \{\lim_{t\to\infty} A(t) < \infty\}$, $\frac{M^{\boldsymbol{\zeta}}(t)}{A(t)}$ converges to an a.s.-finite random variable ξ_1 (thanks to the continuity property of the paths of the Brownian motion). Finally,

$$\limsup_{t\to\infty} \left|\frac{M^{\boldsymbol{\zeta}}(t)}{t}\right| \leq \limsup_{t\to\infty} \frac{M^{\boldsymbol{\zeta}}(t)}{A(t)} \frac{A(t)}{t} = \left\{ \begin{array}{ll} 0 \cdot \xi_0, & \text{on } \mathrm{Tr}_A^{\boldsymbol{\zeta}} \cap \mathrm{Tr}_X^{\boldsymbol{\zeta}} \\ \xi_1 \cdot 0, & \text{on } (\mathrm{Tr}_A^{\boldsymbol{\zeta}})^c \cap \mathrm{Tr}_X^{\boldsymbol{\zeta}} \end{array} \right\} = 0 \text{ on } \mathrm{Tr}_X^{\boldsymbol{\zeta}}.$$

□

Let $\{G(t)\}_{t\in[0,\infty)}$ be the $\mathbb{R}^n$-valued random correspondence defined by

$$G(t) \triangleq \left\{\boldsymbol{\zeta} \in \mathbb{R}^n \,:\, \boldsymbol{\zeta}^T\boldsymbol{\mu}(t) \geq \tfrac{1}{2}||\boldsymbol{\zeta}^T\boldsymbol{\sigma}(t)||^2\right\}. \tag{4.12}$$

The set of all $\boldsymbol{\zeta}(t) \in \mathcal{A}$ with the property that $\boldsymbol{\zeta}(t) \in G(t)$, for all $t \geq 0$, a.s., will be denoted by $\mathcal{A}^G$.

Lemma 4.11. *For each portfolio-proportion process $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)} \in \mathcal{A}^G$, the wealth process $\{X^{\boldsymbol{\zeta}}(t)\}_{t\in[0,\infty)}$ is transient, i.e. $\lim_{t\to\infty} X^{\boldsymbol{\zeta}}(t) = +\infty$, a.s.*

Proof. Pick $\boldsymbol{\zeta}(t) \in \mathcal{A}^G$, let $M^{\boldsymbol{\zeta}}(t)$ be given by (2.8), and $A(t)$ be as in the proof of Lemma 4.10. Following the mentioned proof of Lemma 4.10, we can write $M^{\boldsymbol{\zeta}}(t) = B_{A(t)}$ for some Brownian motion $\{B_t\}_{t\in[0,\infty)}$. Therefore, by Itô's lemma,

$$\log(X^{\boldsymbol{\zeta}}(t)) = \log(X(0)) + \int_0^t Q(u, \boldsymbol{\zeta}(u))\, du + B_{A(t)}.$$

The assumption $\boldsymbol{\zeta}(t) \in \mathcal{A}^G$ implies that

$$Q(t, \boldsymbol{\zeta}(t)) = r + \boldsymbol{\zeta}^T(t)\boldsymbol{\mu}(t) - \tfrac{1}{2}||\boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)|| \geq r, \text{ for all } t > 0, \text{ a.s.,}$$

so the claim of the Lemma will follow once we establish the equality

$$\lim_{t\to\infty} \frac{B_{A(t)}}{t} = 0, \text{ a.s.} \tag{4.13}$$

By Lemma 4.1 combined with the assumption $\boldsymbol{\zeta} \in G(t)$ we have

$$\tfrac{1}{2}||\boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)||^2 \leq \boldsymbol{\zeta}^T(t)\boldsymbol{\mu}(t) \leq ||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||||\boldsymbol{\zeta}^T(t)\boldsymbol{\sigma}(t)||,$$

and so,

$$A(t) = \int_0^t ||\boldsymbol{\zeta}^T(u)\boldsymbol{\sigma}(u)||^2\, du \leq 4\int_0^t ||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||^2\, du, \text{ for all } t \geq 0, \text{ a.s.}$$

By Assumption 2.5, and the inequality (4.14) we see that

$$\limsup_{t\to\infty} \frac{A(t)}{t} < \infty, \text{ for all } t \geq 0, \text{ a.s.}$$

The remainder of the proof of the statement (4.13) parallels the final argument of the proof of Lemma 4.10. □

Lemma 4.12. *Let $\{X(t)\}_{t\in[0,\infty)}$ non-negative process, and let $\mathrm{Tr}_X = \{\lim_{t\to\infty} X(t) = +\infty\} \in \mathcal{F}_\infty$. Then*

$$\lim_{t\to\infty} (\beta(t, X(t)) - \beta^\infty(t)) = 0, \textit{ a.s. on } \mathrm{Tr}_X.$$

Proof. Define the random process $\{\chi(t)\}_{t\in[0,\infty)}$ by

$$\chi(t) = \begin{cases} 0, & \boldsymbol{\zeta}_M(t) \in F(t,\infty), \\ 1, & \boldsymbol{\zeta}_M(t) \in F(t, X(t)) \setminus F(t,\infty), \\ 2, & \boldsymbol{\zeta}_M(t) \in F(t, X(t))^c \end{cases}$$

so that $\beta^{\infty}(t) = 1$ when $\chi(t) = 0$, and $\beta(t, X(t)) = 1$ when $\chi(t) = 0$ or $\chi(t) = 1$. Thus, our task is reduced to the one of establishing the following two claims

$$\lim_{t\to\infty} (\beta(t, X(t)) - \beta^{\infty}(t))\mathbf{1}_{\{\chi(t)=2\}} = 0, \text{ a.s. on } \mathrm{Tr}_X, \text{ and} \tag{4.14}$$

$$\lim_{t\to\infty} (1 - \beta^{\infty}(t))\mathbf{1}_{\{\chi(t)=1\}} = 0, \text{ a.s. on } \mathrm{Tr}_X. \tag{4.15}$$

Claim (4.14)**:** By the estimate (4.5) in Lemma 4.3, for large enough $X(t)$ and $x > X(t)$ we have

$$\mathbf{1}_{\{\chi(t)=2\}}\Big(\beta(t, X(t)) - \beta(t, x)\Big) \leq \mathbf{1}_{\{\chi(t)=2\}}\frac{h(X(t)) - h(x)}{-g(0,0)},$$

and, letting $x \to \infty$ yields,

$$0 \leq \mathbf{1}_{\{\chi(t)=2\}}\Big(\beta(t, X(t)) - \beta^{\infty}(t)\Big) \leq \mathbf{1}_{\{\chi(t)=2\}}\frac{h(X(t))}{-g(0,0)},$$

which, in turn, implies (4.14).

Claim (4.15)**:** Let A' be the set of all $\omega \in A$ for which the limit in (4.15) does not exist or differs from 0. Fix a typical $\omega \in A'$, and pick a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$, $\chi(t_n) = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(1 - \beta_n) \to l > 0$, where $\beta_n = \beta^{\infty}(t_n)$. It is easily seen that $\kappa = \beta_n$ is the unique root of the equation

$$f(\kappa\lambda_n^2, \kappa\lambda_n) = 0, \text{ where } \lambda_n = ||\boldsymbol{\zeta}_M^T(t_n)\boldsymbol{\sigma}(t_n)||.$$

Since $\chi(t) = 1$, we know that $f(\lambda_n^2, \lambda_n) \leq h(X(t_n))$. Thus, $\limsup_n f(\lambda_n^2, \lambda_n) \leq 0$. By joint convexity of f,

$$0 = f(\beta_n\lambda_n^2, \beta_n\lambda) \leq (1 - \beta_n)f(0,0) + \beta_n f(\lambda_n, \lambda_n^2).$$

Passing to the limit we get

$$0 \leq \limsup_n (1 - \beta_n)f(0,0) + \beta_n f(\lambda_n, \lambda_n^2) \leq \lim_n (1 - \beta_n)f(0,0) = lf(0,0).$$

This is in contradiction with the fact that $f(0,0) < 0$, and we can conclude that there is no typical $\omega \in A'$. □

4.5. **Proving optimality.** We are finally ready to show that both $\boldsymbol{\zeta}^*(t)$ and $\boldsymbol{\zeta}^{\infty}(t)$ are optimal. The first step is to identify the (common) value of those strategies. After that we show that no other strategy can produce a higher value.

Lemma 4.13. *There exists a (deterministic) function $\delta : [0,\infty) \to [0,1]$ such that*

$$\beta^*(t) = \delta(||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||).$$

Proof. It is a simple consequence of the regularity properties of the functions f and h that $\beta^*(t)$ can be characterized as $\beta^*(t) = \min(1, g^*(t))$, where $g^*(t) = \kappa(||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||)$ is the unique solution of

$$g(t,\kappa) = f(\kappa||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||^2, \kappa||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||) = 0.$$

Therefore, $\delta(\lambda) = \min(1, \kappa(\lambda))$ is the sought-for function. $\square$

Lemma 4.14.

$$\lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} = \lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^\infty}(t))}{t} = r + Z(x^2\delta(x)), \tag{4.16}$$

where $Z(\cdot)$ is defined in Assumption 2.5, and δ is the function from Lemma 4.13.

Proof. We first show that the limits in (4.16) are equal. It is a matter of a simple calculation to show that both strategies $\{\boldsymbol{\zeta}^*(t)\}_{t\in[0,\infty)}$ and $\{\boldsymbol{\zeta}^\infty(t)\}_{t\in[0,\infty)}$, belong to $\mathcal{A}^G$, where $\mathcal{A}^G$ is introduced after (4.12). Therefore, by Lemmas 4.10 and 4.11,

$$\lim_{t\to\infty} \frac{1}{t}\left(\log X^{\boldsymbol{\zeta}^*}(t) - \log(X^{\boldsymbol{\zeta}^\infty}(t)) - \int_0^t \Big(Q(u,\boldsymbol{\zeta}^*(u)) - Q(u,\boldsymbol{\zeta}^\infty(u))\Big)\,du\right) = 0, \text{ a.s.},$$

so it is enough to show that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \left[Q(u,\boldsymbol{\zeta}^*(u)) - Q(u,\boldsymbol{\zeta}^\infty(u))\right]\,du = 0, \text{ a.s.}$$

Direct computation yields that

$$Q(t,\boldsymbol{\zeta}^*(t)) - Q(t,\boldsymbol{\zeta}^\infty(t)) = (\beta^*(t) - \beta^\infty(t))\left(1 - \frac{1}{2}(\beta^*(t) + \beta^\infty(t))\right)\boldsymbol{\zeta}_M^T(t)\boldsymbol{\mu}(t),$$

so thanks to the ergodic property of the process $\{\boldsymbol{\zeta}_M^T(t)\boldsymbol{\mu}(t)\}_{t\in[0,\infty)}$ (Assumption 2.5), it will be enough to show that $\lim_{t\to\infty}(\beta^*(t) - \beta^\infty(t)) = 0$, a.s. This, however, follows from Lemma 4.12.

To identify the limit, we use the Lemma 4.10 to conclude that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}^*}(t))}{t} &= r + \lim_{t\to\infty}\frac{1}{t}\int_0^t \beta^*(u)\boldsymbol{\zeta}_M^T(u)\boldsymbol{\mu}(u)\,du \\
&= r + \lim_{t\to\infty}\frac{1}{t}\int_0^t \delta(||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||)\boldsymbol{\zeta}_M^T(u)\boldsymbol{\mu}(u)\,du \\
&= r + \lim_{t\to\infty}\frac{1}{t}\int_0^t \delta(||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||)||\boldsymbol{\zeta}_M^T(u)\,\boldsymbol{\sigma}(u)||^2\,du = r + Z(x^2\delta(x)).
\end{aligned}$$

$\square$

Lemma 4.15. *For each $\{\boldsymbol{\zeta}(t)\}_{t\in[0,\infty)} \in \mathcal{A}$ we have*

$$\liminf_{t\to\infty} \frac{\log(X^{\boldsymbol{\zeta}}(t))}{t} = \begin{cases} 0, & \text{on } \left\{\liminf_{t\to\infty} X^{\boldsymbol{\zeta}}(t) < \infty\right\} \\ \liminf_{t\to\infty} \frac{1}{t}\int_0^t \tilde{Q}(t,\boldsymbol{\zeta}(t))\,dt, & \text{on } \left\{\lim_{t\to\infty} X^{\boldsymbol{\zeta}}(t) = \infty\right\} \end{cases}$$

Proof. Itô's formula applied to the process $\{\log(X^{\boldsymbol{\zeta}}(t))\}_{t\in[0,\infty)}$ yields

$$\frac{1}{t}\log(X^{\boldsymbol{\zeta}}(t)) = \frac{X(0)}{t} + \frac{1}{t}\int_0^t \tilde{Q}(u,\boldsymbol{\zeta}(u))\,du + \frac{1}{t}\int_0^t \boldsymbol{\zeta}^T(u)\boldsymbol{\sigma}(u)\,d\boldsymbol{W}(u).$$

it remains to to let $t\to\infty$ and apply the result of Lemma 4.10. □

Theorem 4.16. *The portfolio-proportion process $\boldsymbol{\zeta}^{\infty}(t)$ is optimal, i.e.,*

$$\liminf_{t\to\infty}\frac{1}{t}\log(X^{\boldsymbol{\zeta}}(t)) \leq \lim_{t\to\infty}\frac{1}{t}\log(X^{\boldsymbol{\zeta}^{\infty}}(t)) = r + Z(x^2\delta(x^2)),\ \textit{a.s., for each } \boldsymbol{\zeta}(t)\in\mathcal{A},$$

where δ is the function introduced in Lemma 4.13.

Proof. Pick $\boldsymbol{\zeta}(t)\in\mathcal{A}$ and recall that, by Lemma 4.15 and strict positivity of the parameter r, it will be enough to show that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t Q(u,\boldsymbol{\zeta}(u))\,du \leq \liminf_{t\to\infty}\frac{1}{t}\int_0^t Q(u,\boldsymbol{\zeta}^{\infty}(u))\,du,\ \text{on } \mathrm{Tr}_X^{\boldsymbol{\zeta}} \triangleq \left\{\lim_{t\to\infty} X^{\boldsymbol{\zeta}}(t) = +\infty\right\}.$$

Let $d_{\boldsymbol{\sigma}(t)}$ is the metric on $\mathbb{R}^n$ defined in (2.11) so that

$$Q(t,\boldsymbol{\zeta}) = r + \boldsymbol{\zeta}^T\boldsymbol{\mu} - \tfrac{1}{2}||\boldsymbol{\zeta}^T\boldsymbol{\sigma}|| = (r + \tfrac{1}{2}||\boldsymbol{\zeta}_M^T\boldsymbol{\sigma}||^2) - \tfrac{1}{2}d^2_{\boldsymbol{\sigma}}(\boldsymbol{\zeta},\boldsymbol{\zeta}_M). \tag{4.17}$$

Furthermore, we have the following simple expression

$$Q(t,\boldsymbol{\zeta}^{\infty}(t)) - Q(t,\boldsymbol{\zeta}(t)) = \tfrac{1}{2}\Big(d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}(t),\boldsymbol{\zeta}_M(t)) - d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}^{\infty}(t),\boldsymbol{\zeta}_M(t))\Big). \tag{4.18}$$

Consequently, all we need to show is the following inequality

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \left[d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}(u),\boldsymbol{\zeta}_M(t)) - d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}^{\infty}(u),\boldsymbol{\zeta}_M(u))\right]du \geq 0,\ \text{a.s. on } \mathrm{Tr}_X^{\boldsymbol{\zeta}}. \tag{4.19}$$

Being an element of $F(t,X^{\boldsymbol{\zeta}}(t))$, the vector $\boldsymbol{\zeta}(t)$ is $d_{\boldsymbol{\sigma}(t)}$-further away from $\boldsymbol{\zeta}_M(t)$ than the projection $\beta(t,X^{\boldsymbol{\zeta}}(t))\boldsymbol{\zeta}_M(t)$ of $\boldsymbol{\zeta}_M(t)$ onto $F(t,X^{\boldsymbol{\zeta}}(t))$. Therefore, the expression inside the $\limsup$ in (4.19) dominates the difference $d^2_{\boldsymbol{\sigma}(t)}(\beta(t,X^{\boldsymbol{\zeta}}(t))\boldsymbol{\zeta}_M(u),\boldsymbol{\zeta}_M(t)) - d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}^{\infty}(u),\boldsymbol{\zeta}_M(u))$ of squared distances. Furthermore, this difference can be rewritten as

$$(\beta(t,X^{\boldsymbol{\zeta}}(t)) - \beta^*(t))^2||\boldsymbol{\zeta}_M^T(t)\boldsymbol{\sigma}(t)||.$$

It remains to employ the ergodicity Assumption 2.5, and use the result of Lemma 4.12, which states that $\beta(t,X^{\boldsymbol{\zeta}}(t)) - \beta^{\infty}(t)\to 0$ on $\mathrm{Tr}_X^{\boldsymbol{\zeta}}$. □

4.6. **The relative constraints.** We deal with the relative constraints in this last subsection. The infinite-horizon ergodic optimization problem can be treated in a fashion virtually identical to the case of absolute constraints, so we leave it to the interested reader.

It remains to deal with the finite-horizon problem of optimal expected logarithmic utility. As before, $\mathcal{A}$ will denote a generic admissibility set corresponding to a pair (f, h) of functions satisfying the assumptions in Definition 2.9 (with the variant (A) for the function h). Moreover, we pick a time horizon $T > 0$.

Define the process $\{\boldsymbol{\zeta}^r(t)\}_{t\in[0,\infty)}$ as a $d_{\boldsymbol{\sigma}(t)}$-projection of $\boldsymbol{\zeta}_M(t)$ onto the instantaneous constraint set $F(t)$.

Lemma 4.17. *For any $\boldsymbol{\zeta} \in F(t)$, the following inequality holds*

$$d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \boldsymbol{\zeta}) \geq d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \boldsymbol{\zeta}^r(t)) + d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}^r(t), \boldsymbol{\zeta}). \tag{4.20}$$

Proof. $\boldsymbol{\zeta}^r(t)$ is defined as the minimizer (in the convex set $F(t)$) of the distance $\gamma(\cdot) = d_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \cdot)$. Therefore the directional derivative of the square $\gamma^2(\cdot)$ in the direction $\boldsymbol{\zeta} - \boldsymbol{\zeta}^r(t)$, evaluated at the point $\boldsymbol{\zeta}^r(t)$, must be non-positive, i.e.,

$$\begin{aligned} 0 &\geq \nabla\gamma^2(\boldsymbol{\zeta}^r(t))(\boldsymbol{\zeta} - \boldsymbol{\zeta}^r(t)) = (\boldsymbol{\zeta}^r(t) - \boldsymbol{\zeta}_M(t))^T \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)^T(\boldsymbol{\zeta}^r(t) - \boldsymbol{\zeta}) \\ &= \tfrac{1}{2}\Big(d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \boldsymbol{\zeta}^r(t)) + d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}^r(t), \boldsymbol{\zeta}) - d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \boldsymbol{\zeta})\Big). \end{aligned} \tag{4.21}$$

□

Lemma 4.18. *The process $\boldsymbol{\zeta}^r(t)$ is in $\mathcal{A}$ and the quotient*

$$Y^r(t) = \frac{X^{\boldsymbol{\zeta}}(t)}{X^{\boldsymbol{\zeta}^r}(t)}, \ t \in [0, \infty) \tag{4.22}$$

is a strictly positive supermartingale for each $\boldsymbol{\zeta}(t) \in \mathcal{A}$.

Proof. That $\boldsymbol{\zeta}^r(t) \in \mathcal{A}$ follows directly from its construction as a projection onto the instantaneous constraint set $F(t)$. Consequently, both $X^{\boldsymbol{\zeta}}(t)$ and $X^{\boldsymbol{\zeta}^r}(t)$ are strictly positive processes, and therefore, so is Y^r. In order to show that Y^r is a supermartingale, we use the Itô's lemma and expression (4.17) to conclude that its semimartingale decomposition of $Y^r(t)$ is of the form

$$dY^r(t) = -\tfrac{1}{2}\Big(d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \boldsymbol{\zeta}(t)) - d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}_M(t), \boldsymbol{\zeta}^r(t)) - d^2_{\boldsymbol{\sigma}(t)}(\boldsymbol{\zeta}^r(t), \boldsymbol{\zeta}(t))\Big)\, dt + dL_t,$$

where $L(t)$ is a local martingale. By Lemma 4.17, $Y^r(t)$ is a local supermartingale, and its non-negativity allows us to use the standard argument based on the Fatou Lemma to conclude that it is a (true) supermartingale. □

Lemma 4.19. *Let $\bar{\tau}$ be an $[0,\infty)$-valued stopping time. Then*

$$\mathbb{E}[\log(X^{\boldsymbol{\zeta}^r}(\bar{\tau})] \leq \mathbb{E}[\log(X^{\boldsymbol{\zeta}}(\bar{\tau}))],$$

for any $\boldsymbol{\zeta}(t) \in \mathcal{A}$.

Proof. By concavity of the function $\log(\cdot)$, we have

$$\mathbb{E}[\log(X^{\boldsymbol{\zeta}^r}(\bar{\tau})) - \log(X^{\boldsymbol{\zeta}}(\bar{\tau}))] \geq \mathbb{E}[(X^{\boldsymbol{\zeta}^r}(\bar{\tau}) - X^{\boldsymbol{\zeta}}(\bar{\tau}))\frac{1}{X^{\boldsymbol{\zeta}^r}(\bar{\tau})}] = 1 - \mathbb{E}[\frac{X^{\boldsymbol{\zeta}}(\bar{\tau})}{X^{\boldsymbol{\zeta}^r}(\bar{\tau})}] \leq 0,$$

where the last inequality follows from Lemma 4.22 and the optional sampling theorem. □

Appendix A. Some technical results

Proof of Proposition 2.16. The expression (2.18) for VaR follows directly from its definition. In the case of TVaR, the conditional expectation in (2.17) can be written as

$$\mathrm{TVaR}(x,\zeta_{\boldsymbol{\mu}},\zeta_{\boldsymbol{\sigma}}) = \left(\frac{x}{\alpha\sqrt{2\pi}}\int_{-\infty}^{N^{-1}(\alpha)} \left[1-\exp\left\{Q(\zeta_{\boldsymbol{\mu}},\zeta_{\boldsymbol{\sigma}})\tau + y\zeta_{\boldsymbol{\sigma}}\sqrt{\tau}\right\}\right] e^{-\frac{y^2}{2}}\, dy\right)^+. \tag{A.1}$$

This integral readily evaluates to (2.19). Finally, the calculation of LEL is identical to the one for TVaR with $\boldsymbol{\mu} = 0$. □

Compliance with Definition 2.9. We only concentrate on the absolute case, as the relative one is completely analogous and easier. For the

VaR**-constraint:** Take

$$f_V^{\mathrm{abs}}(\zeta_{\boldsymbol{\mu}},\zeta_{\boldsymbol{\sigma}}) = -\tau(r+\zeta_{\boldsymbol{\mu}} - \tfrac{1}{2}\zeta_{\boldsymbol{\sigma}}^2) - N^{-1}(\alpha)\zeta_{\boldsymbol{\sigma}}\sqrt{\tau}, \quad h_V^{\mathrm{abs}}(x) = -\log\left[(1-\frac{a_V^{\mathrm{abs}}}{x})^+\right].$$

All of the properties (1)-(5) from the statement of the Proposition can be obtained easily.

TVaR**-constraint:** Set

$$f_T^{\mathrm{abs}}(\zeta_{\boldsymbol{\mu}},\zeta_{\boldsymbol{\sigma}}) = \log(\alpha) - \tau(r+\zeta_{\boldsymbol{\mu}}) - \log(N(N^{-1}(\alpha) - \zeta_{\boldsymbol{\sigma}}\sqrt{\tau})), \quad h_T^{\mathrm{abs}}(x) = -\log(1-\frac{a_T^{\mathrm{abs}}}{x})^+. \tag{A.2}$$

For the TVaR case, we only discuss the estimate (2.14), while the other properties follow simply from (A.2). For (2.14) we simply note that

$$\lim_{\zeta_{\boldsymbol{\sigma}}\to\infty} \frac{\log(N(N^{-1}(\alpha)-\zeta_{\boldsymbol{\sigma}}\sqrt{\tau}))}{\zeta_{\boldsymbol{\sigma}}^2} = -\tfrac{1}{2}\tau.$$

LEL**-constraint:** LEL is a special case of TVaR with $\zeta_{\boldsymbol{\mu}} = 0$.

□

References

[AC88] P.H. Algoet and T.M. Cover, *Asymptotic optimality and asymptotic equipartion properties of log-optimum investments*, Ann. Appl. Probab. (1988), no. 16, 876–898.

[AØ88] K. Aase and B. Øksendal, *Admissible investment strategies in continuous trading.*, Stoch. Proc. Appl. (1988), no. 30, 291–301.

[ADEH] Artzner, P., Delbaen, F., Eber, J.M. & Heath, D. (1999) Coherent measures of risk, *Math. Finance* **9**, 203–228.

[ADEHK02] Artzner, P, Delbaen, F., Eber, J.-M., Heath, D. & Ku, H. (2002) Coherent multiperiod risk measurement, working paper, Department of Mathematics, ETH-Zürich.

[ADEHK04] Artzner, P, Delbaen, F., Eber, J.-M., Heath, D. & Ku, H. (2004) Coherent multiperiod risk adjusted values and Bellman's principle, forthcoming in Ann. Oper. Res.

[AP05] C. Atkinson and M. Papakokinou, *Theory of optimal consumptionand portfolio selection under a capital-at-risk (car) and a value-at-risk (var) constraint,*, IMA Journal of Management Mathematics (2005), no. 16, 37–70.

[AST01] M. Akian, A. Sulem, and M. I. Taksar, *Dynamic optimization of long-term growth rate for a portfolio with transaction costs and logarithmic utility*, Math. Finance **11** (2001), no. 2, 153–188. MR 2001k:91064

[BCK05] A. Berkelaar, P. Cumperayot, and R. Kouwenberg, *The effect of var-based risk management on asset prices and volatility smile*, Europen Financial Management (2005), no. 8, 65–78.

[Bel57] R. Bellman, *A Markovian decision process*, J. Math. Mech (1957), no. 6, 679–684.

[Bre61] L. Breiman, *Optimal gambling systems for favorable games*, 4th Berkeley Symposium on Probability and Statistics **1** (1961), 65–78.

[BS01] S. Basak and A. Shapiro, *Value-at-risk-based risk management: optimal policies and asset prices,*, Rev. Financial Studies (2001), no. 14, 371–405.

[DCK04] F. Delbaen and P. Cheridito and M. Kupper, *Coherent and convex monetary risk measures for bounded cdlg processes*, Stochastic Processes and their Applications (2004), no. 112, vol. 1, 2004, 1–22.

[CHI07] D. Cuoco, H. He, and S Issaenko, *Optimal dynamic trading strategies with risk limits*, Operations Research (2007), to appear.

[CK85] R. M. Cox and I. Karatzas, *Stationary control of brownian motion in several dimensions*, Adv. Appl. Probab (1985), no. 17, 531–561.

[CK95] J. Cvitanić and I. Karatzas, *On portfolio optimization under "drawdown" constraints*, IMA Volumes in Math. and its Appl., no. 65, 1995, pp. 35–46.

[CL03] D. Cuoco and H. Liu, *An analysis of VaR-based capital requirements*, Journal of Financial Intermediation, no. 15, 2006, pp. 362–394.

[Cov84] T. M. Cover, *An algorithm for maximizing expected log-investment return*, IEEE Trans. Inform. Theory (1984), no. 30, 369–373.

[Cov91] ———, *Universal portfolios*, Mathematical Finance (1991), no. 1, 1–29.

[DVLLLW03] G. Dmitrasinović-Vidović, A. Lari-Lavassani, X. Li, and T. Ware, *Dynamic portfolio selection under capital at risk*, University of Calgary Yellow Series, Report 833, 2003.

[ESR01] C. Klüppelberg, S. Emmer. and R. Korn , *Optimal portfolios with bounded capital at risk*, Mathematical Finance (2001), no. 11, 365–384.

[FHH03] W. Fleming and D. Hernández-Hernández, *An optimal consumption model with stochastic volatility*, Finance and Stochastics (2003), no. 7, 245–262.

[FS99] W. H Fleming and S. J. Sheu, *Optimal long term rate of expected utility of wealth*, Ann. Appl. Probab. (1999), no. 9, 871–903.

[FS00] W. H. Fleming and S. J. Sheu, *Risk sensitive control and an optimal investment model*, Mathematical Finance (2000), no. 10, 197–213.

[FS04] W. H Fleming and S. J. Sheu, *Risk sensitive control and optimal investment model ii*, 2004.

[FS04a] Föllmer, H. & Schied, A. *Stochastic Finance*, second ed., de Gruyter, Berlin, 2004.

[FT02] J. P. Fouque and T. Tullie, *Variance reduction for Monte Carlo simulation in a stochastic volatility environment*, Quantitative Finance (2002), no. 2, 24–30.

[GGW05] A. Gabih, W. Grecksch and R. Wunderlich, *Dynamic portfolio optimization with bounded shortfall risks*, Stochastic Analysis and Applications (2005), no. 23, 579–594.

[GW06] A. Gundel and S. Weber, *Robust utility maximization with limited downside risk in incomplete markets*, Tech. report, Cornell University, 2006, http://legacy.orie.cornell.edu∼sweber/GuWe2006.pdf.

[GZ93] S. Grossman and Z. Zhou, *Optimal investment strategies for controlling drawdowns*, Math. Finance (1993), no. 3, 241–276.

[Hak70] N. Hakansson, *Optimal investment and consumption strategies under risk for a class of utility functions*, Econometrica (1970), no. 38, 587–607.

[JK01] Jaschke, S. & Küchler, U. (2001) Coherent risk measures and good-deal bounds, *Finance Stochastics* **5**, 181–200.

[Jam91] F. Jamshidian, *Asymptotically optimal portfolios*, Math Finance (1991), no. 2, 131–150.

[JO97] Jorion, P. (1997) *Value at Risk*, Irwin, Chicago.

[Kar89] I. Karatzas, *Optimization problems in the theory of continuous trading*, SIAM J. Control Optim. **27** (1989), no. 6, 1221–1259.

[Kel56] J.L. Kelly, *A new interpretation of information rate*, Bell System Techn. Journal **35** (1956), 917–926.

[Kha60] R. Z. Khasminskii, *Ergodic propertis of recurrent diffusion processes and stabilization of the solution to the cauchy problem for parabolic equations,*, Theory of Probabilities and Its Applications **5** (1960), no. 2, 179–196.

[KS91] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991.

[KS98] ———, *Methods of mathematical finance*, Applications of Mathematics (New York), vol. 39, Springer-Verlag, New York, 1998.

[Las74] J. M. Lasry, *Controle stochastique ergodique*, Doctoral dissertation, 1974.

[LVT02] M. Leippold, P. Vanini and F. Trojani,, *Equilibrium impact of value-at-risk*, 2002.

[Mer69] R. C. Merton, *Lifetime portfolio selection under uncertainty: the continuous-time case*, Rev. Econom. Statist. **51** (1969), 247–257.

[Mer71] ———, *Optimum consumption and portfolio rules in a conitinuous-time model*, J. Economic Theory **3** (1971), 373–413.

[MSZZ04] L. C. MacLean, R. Sanegre, Y. Zhao, and W. T. Ziemba, *Capital growth with security*, J. Econom. Dynam. Control **28** (2004), no. 5, 937–954, Financial decision models in a dynamical setting.

[Pro04] P. E. Protter, *Stochastic integration and differential equations*, second ed., Applications of Mathematics (New York), vol. 21, Springer-Verlag, Berlin, 2004, Stochastic Modelling and Applied Probability.

[PS94] S. Pliska and M. Selby, *On a free boundary problem that arises in portfolio management*, Phil. Trans. Roy. Soc. London, Ser. A (1994), no. 347, 555–561.

[Rob83] M. Robin, *Long-term average cost control problems*, Acta. Appl. Math. (1983), no. 1, 281–300.

[Sam79] P. A. Samuelson, *Why we should not make mean log of wealth big though years to act are long*, Journal of Banking and Finance (1979), no. 3, 305–307.

[SS91] E. M. Stein and J. C. Stein, *Stock Price Distributions with Stochastic Volatility: An Analytic Approach*, The Review of Financial Studies (1991), vol. 4, no. 4, 727–752.

[Tar82] R. Tares, *Comportement asymptotique d'un probleème de control stochastique,*, Cashier de math. de la Decision, no. 8215, 1982.

[Tar85] ______, *Asymptotic evolution of stochastic control problem*, SIAM Journal of Control and Optimization (1985), no. 23, 614–631.

[Tho71] E. O. Thorp, *Portfolio choice and the kelly criterion*, Stochastic Models in Finance, 1971, pp. 599–619.

[TKA88] M. Taksar, M. J. Klass, and D. Assaf, *A diffussion model for optimal portfolio selection in the presence of brokerage fees*, Math. Operations Reserch (1988), no. 13, 277–294.

[Yiu04] K. F. C. Yiu, *Optimal portfolios under a value-at-risk constraint*, Journal of Economic Dynamics& Control (2004), no. 28, 1317–1334.

[W03] T. Wang, *Conditional preferences and updating*, J. Econom. Theory (2003), no. 108, 286–321.